Limits, Continuity & Derivability

Calculus required continuity, and continuity was supposed to require the infinitely little; But nobody could discover what the infinitely little might be............Russell, Bertrand

Definition : Limit of a function f(x) is said to exist, as $x \to a$ when,

$$\underset{h \to 0^+}{\ell im} f(a - h) = \underset{h \to 0^+}{\ell im} f(a + h) = \text{Finite}$$

(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at x = a. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '$-\infty$', then it is said to be infinite limit.

Remember, '$x \to a$' means that x is approaching to 'a' but not equal to 'a'.

Fundamental theorems on limits :

Let $\underset{x \to a}{\ell im} f(x) =$ and $\underset{x \to a}{\ell im} g(x) = m$. If ℓ & m are finite, then:

(A) $\underset{x \to a}{\ell im} \{ f(x) \pm g(x) \} = \ell \pm m$

(B) $\underset{x \to a}{\ell im} \{ f(x).\, g(x) \} = \ell . m$

(C) $\underset{x \to a}{\ell im} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$, provided $m \neq 0$

(D) $\underset{x \to a}{\ell im} k f(x) = k \underset{x \to a}{\ell im} f(x) = k\ell$; where k is a constant.

(E) $\underset{x \to a}{\ell im} f(g(x)) = f\left(\underset{x \to a}{\ell im} g(x)\right) = f(m)$; provided f is continuous at $g(x) = m$.

Example # 1 : Evaluate the following limits : -

(i) $\underset{x \to 2}{\ell im} (x + 2)$ (ii) $\underset{x \to 0}{\ell im} \cos(\sin x)$

Solution : (i) x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\underset{x \to 2}{\ell im} (x + 2) = 2 + 2 = 4$

(ii) $\underset{x \to 0}{\ell im} \cos(\sin x) = \cos\left(\underset{x \to 0}{\ell im} \sin x\right) = \cos 0 = 1$

Self practice problems

Evaluate the following limits : -

(1) $\underset{x \to 2}{\ell im} x(x - 1)$ (2) $\underset{x \to 2}{\ell im} \dfrac{x^2 + 4}{x + 2}$

Ans. (1) 2 (2) 2

Indeterminate forms :

If on putting x = a in f(x), any one of $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\mathbf{0 \times \infty, \infty - \infty, \infty^0, 0^0, 1^\infty}$ form is obtained, then the limit has an indeterminate form. All the above forms are interchangeable, i.e. we can change one form to other by suitable substitutions etc.

In such cases $\underset{x \to a}{\ell im} f(x)$ may exist.

Consider $f(x) = \dfrac{x^2 - 4}{x - 2}$. Here $\lim\limits_{x\to 2} x^2 - 4 = 0$ and $\lim\limits_{x\to 2} x - 2 = 0$

$\therefore$ $\lim\limits_{x\to 2} f(x)$ has an indeterminate form of the type $\dfrac{0}{0}$

$\lim\limits_{x\to\infty} \dfrac{\ell nx}{x}$ has an indeterminate form of type .

$\lim\limits_{x\to 0} (1 + x)^{1/x}$ is an indeterminate form of the type 1^{∞}

NOTE :

(i) $\infty + \infty = \infty$

(ii) $\infty \times \infty = \infty$

(iii) $\dfrac{a}{\infty} = 0$, if a is finite.

(iv) is not defined for any $a \in R$.

(v) $\lim\limits_{x\to 0} \dfrac{x}{x}$ is an indeterminate form whereas $\lim\limits_{x\to 0} \dfrac{[x^2]}{x^2}$ is not an indeterminate form

(where [.] represents greatest integer function) Students may remember these forms along with the prefix 'tending to' i.e. $\dfrac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form where as $\dfrac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

Similarly (tending to one)$^{\text{tending to } \infty}$ is indeterminate form whereas (exactly one)$^{\text{tending to } \infty}$ is not an indeterminate form, its value is one.

To evaluate a limit, we must always put the value where 'x' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminate form comes, we have to remove the indeterminancy, once the indeterminancy is removed the limit can be evaluated by putting the value of x, where it is approaching.

Methods of removing indeterminancy

Basic methods of removing indeterminancy are

(A) Factorisation (B) Rationalisation
(C) Using standard limits (D) Substitution
(E) Expansion of functions.

Factorisation method : -

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Example # 2 $\lim\limits_{x\to 3} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$

Solution : $\lim\limits_{x\to 3} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim\limits_{x\to 3} \dfrac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

Rationalisation method :-

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Example # 3 : Evaluate :

(i) $\lim\limits_{x\to 1} \dfrac{3 - \sqrt{8x+1}}{5 - \sqrt{24x+1}}$ (ii) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+x} - \sqrt{1-x}}$

Solution : (i) $\lim\limits_{x\to 1} \dfrac{3-\sqrt{8x+1}}{5-\sqrt{24x+1}} = \lim\limits_{x\to 1} \dfrac{(9-8x-1)\left(5+\sqrt{24x+1}\right)}{\left(3+\sqrt{8x+1}\right)(25-24x-1)} = \dfrac{5}{9}$

(ii) The form of the given limit is when $x \to 0$. Rationalising the numerator, we get

$$\lim_{x\to 1} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} = \lim_{x\to 1}\left[\frac{x}{\sqrt{1+x}-\sqrt{1-x}} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 1}\left[\frac{x\ (\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)}\right] = \lim_{x\to 1}\left[\frac{x\ (\sqrt{1+x}+\sqrt{1-x})}{2x}\right] = \lim_{x\to 1}\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right] = \frac{2}{2} = 1$$

Self practice problems

Evaluate the following limits : -

(3) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{1-(\sin x)^{1/3}}{1-(\sin x)^{2/3}}$ (4) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

(5) $\lim\limits_{x\to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ (6) $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}}$

Ans. (3) $\dfrac{1}{2}$ (4) $\dfrac{1}{2\sqrt{x}}$ (5) $\dfrac{1}{4a\sqrt{a-b}}$ (6) 0

Standard limits :

(a) (i) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$ [Where x is measured in radians]

(ii) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1$

(iii) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$; $\lim\limits_{x\to 0} (1+ax)^{\frac{1}{x}} = e^a$

(iv) $\lim\limits_{x\to 0} \left(1+\dfrac{1}{x}\right)^x = e$; $\lim\limits_{x\to 0} \lim\limits_{x\to\infty} \left(1+\dfrac{a}{x}\right)^x = e^a$

(v) $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1$; $\lim\limits_{x\to 0} \dfrac{a^x-1}{x} = \log_e a = \ell n a$,a > 0

(vi) $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)}{x} = 1$

(vii) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}$

(b) If $f(x) \to 0$, when $x \to a$, then

(i) $\lim\limits_{x\to a} \dfrac{\sin f(x)}{f(x)} = 1$ (ii) $\lim\limits_{x\to a} \cos f(x) = 1$

(iii) $\lim\limits_{x\to a} \dfrac{\tan f(x)}{f(x)} = 1$ (iv) $\lim\limits_{x\to a} \dfrac{e^{f(x)}-1}{f(x)} = 1$

(v) $\lim\limits_{x\to a} \dfrac{b^{f(x)}-1}{f(x)} = \ell n\, b$, $(b > 0)$ (vi) $\lim\limits_{x\to a} \dfrac{\ell n(1+f(x))}{f(x)} = 1$

(vii) $\lim\limits_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

(c) $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} \phi(x) = B$(a finite quantity), then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = A^B$.

Example # 4 : Evaluate : $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

Solution : $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to a} \frac{(1+x)^n - 1}{(1+x) - 1} = n$

Example # 5 : Evaluate : $\lim_{x\to 0} \frac{3^x - 1}{2^x - 1}$

Solution : $\lim_{x\to 0} \frac{3^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{3^x - 1}{x} . \frac{1}{\frac{2^x - 1}{x}} = \frac{\ell n 3}{\ell n 2}$

Example # 6 : Evaluate : $\lim_{x\to 0} \frac{1-\cos 3x}{x^2}$

Solution : $\lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \lim_{x\to 0} \frac{1}{2} . \left(\frac{3\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2 = \frac{9}{2}$

Example # 7 : Evaluate : $\lim_{x\to 0} \frac{\sin^2 x}{\sin 4x . \tan x}$

Solution : $\lim_{x\to 0} \frac{\sin^2 x}{\sin 4x . \tan x} = \lim_{x\to 0} \frac{x^2\left(\frac{\sin x}{x}\right)^2}{4x\left(\frac{\sin 4x}{4x}\right)\left(\frac{\tan x}{x}\right)x} = \frac{1}{4}$

Example # 8 : Evaluate : $\lim_{x\to 0} (1 + 2x)^{1/x}$

Solution : $\lim_{x\to 0} (1 + 2x)^{1/x} = e^{\lim_{x\to 0} \frac{2}{x}.x} = e^2$.

Example # 9 : Evaluate :

(i) $\lim_{x\to y} \frac{e^x - e^y}{x - y}$ (ii) $\lim_{x\to 0} \frac{x(e^x - 1)}{1-\cos x}$

Solution : (i) $\lim_{x\to y} \frac{e^x - e^y}{x - y} = \lim_{x\to y} \frac{e^y(e^{x-y} - 1)}{x - y} = e^y$

(ii) $\lim_{x\to 0} \frac{x(e^x - 1)}{1-\cos x} = \lim_{x\to 0} \frac{x(e^x - 1)}{2\sin^2\frac{x}{2}} = \frac{1}{2} . \lim_{x\to 0} \left[\frac{e^x - 1}{x} . \frac{x^2}{\sin^2\frac{x}{2}}\right] = 2$

Self practice problems

Evaluate the following limits : -

(7) $\lim_{x\to 0} \frac{\sin 7x}{3x}$

(8) $\lim_{x\to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$

(9) $\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$

(10) $\lim_{x\to 0} \frac{5^x - 9^x}{x}$

(11) $\lim_{x\to \infty} (1 + a^2)^x \sin \frac{b}{(1+a^2)^x}$, where $a \neq 0$

Ans. (7) $\frac{7}{3}$ (8) $\frac{1}{32}$ (9) does not exist (10) $\ell n \frac{5}{9}$ (11) b

Use of substitution in solving limit problems

Sometimes in solving limit problem we convert $\lim\limits_{x\to a} f(x)$ into $\lim\limits_{h\to 0} f(a+h)$ or $\lim\limits_{h\to 0} f(a-h)$ according as need of the problem. (here h is approaching to zero.)

Example # 10 : Evaluate : $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x-\sin x}{\cos x\left(\sqrt{2}-2\sin x\right)}$

Solution : $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x-\sin x}{\cos x\left(\sqrt{2}-2\sin x\right)} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{1}{\sqrt{2}}.\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$ Put $x = \dfrac{\pi}{4} + h$

$\because \quad x \to \dfrac{\pi}{4} \Rightarrow h \to 0$

$$\lim_{h\to 0} \frac{1}{\sqrt{2}} \frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)} = \lim_{h\to 0} \frac{1}{\sqrt{2}} \frac{1-\frac{1+\tan h}{1-\tan h}}{1-\sin h-\cos h} = \lim_{h\to 0} \frac{1}{\sqrt{2}} \frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{2}} \frac{-2\tan h}{2\sin\frac{h}{2}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)} = \lim_{h\to 0} \frac{1}{\sqrt{2}} \frac{-2\frac{\tanh}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tanh)}$$

$$= \frac{-2}{-1}.\frac{1}{\sqrt{2}} = \sqrt{2}$$

Limits using expansion

(a) $a^x = 1+\dfrac{x\ \ell n\ a}{1!}+\dfrac{x^2\,\ell n^2 a}{2!}+\dfrac{x^3\,\ell n^3 a}{3!}+\dots\dots\dots, \quad a > 0$

(b) $e^x = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dots\dots$

(c) $\ell n\,(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dots\dots\dots, \quad \text{for } -1 < x \le 1$

(d) $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dots\dots$

(e) $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dots\dots$

(f) $\tan x = x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\dots\dots$

(g) for $|x| < 1$, $n \in R$; $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{1\ .\ 2}x^2 + \dfrac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}x^3 + \dots\dots\dots\dots\infty$

(h) $(1+x)^{\frac{1}{x}} = e\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-\dots\dots\dots\dots\right)$

Example # 11: Evaluate : $\lim\limits_{x\to 0} \dfrac{\tan x-x}{x^3}$

Solution : $\lim\limits_{x\to 0} \dfrac{\tan x-x}{x^3} = \lim\limits_{x\to 0} \dfrac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots-x}{x^3} = \dfrac{1}{3}$

Example # 12 : Evaluate : $\lim\limits_{x\to 2} \dfrac{(14+x)^{\frac{1}{4}}-2}{x-2}$

Solution : $\lim\limits_{x\to 2}\dfrac{(14+x)^{\frac{1}{4}}-2}{x-2} = \lim\limits_{x\to 2}\dfrac{(14+x)^{\frac{1}{4}}-16^{\frac{1}{4}}}{(14+x)-16} = \dfrac{1}{4}\ .\ 16^{\frac{1}{4}-1} = \dfrac{1}{32}$

Example # 13 : If $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)+\alpha\sin x+\dfrac{x^2}{2}}{x\tan^2 x} = \dfrac{1}{2}$ then find α.

Solution : $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)+\alpha\sin x+\dfrac{x^2}{2}}{x\tan^2 x} = \lim\limits_{x\to 0}\dfrac{\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-.....\right)+\alpha\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-.....\right)+\dfrac{x^2}{2}}{x^3.\dfrac{\tan^2 x}{x^2}} = \dfrac{1}{2}$

$\Rightarrow \qquad \alpha = -1$

Example # 14 : Evaluate : $\lim\limits_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}$

Solution : $\lim\limits_{x\to 0}\dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x} = \lim\limits_{x\to 0}\dfrac{e-e\left(1-\dfrac{x}{2}+......\right)}{\tan x} = \lim\limits_{x\to 0}\ \times\dfrac{x}{\tan x} = \dfrac{e}{2}$

Example # 15 : Find the values of $\alpha + 2\beta + 3\gamma$ if $\lim\limits_{x\to 0}\dfrac{\alpha e^x-\beta\cos x+\gamma e^{-x}}{x\tan x} = 2$

Solution : $\lim\limits_{x\to 0}\dfrac{\alpha e^x-\beta\ \ \cos x+\gamma e^{-x}}{x\tan x} = 2$ .....(1)

at $x \to 0$ numerator must be equal to zero

$\therefore \qquad \alpha - \beta + \gamma = 0 \quad \Rightarrow \quad \beta = \alpha + \gamma$ .....(2)

From (1) & (2), $\lim\limits_{x\to 0}\dfrac{\alpha e^x-(\alpha+\gamma)\ \ \cos x+\gamma e^{-x}}{x^2\dfrac{\tan x}{x}} = 2$

$\Rightarrow \lim\limits_{x\to 0}\dfrac{\alpha\left(1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+......\right)-(\alpha+\gamma)\left(1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-......\right)+\gamma\left(1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+......\right)}{x^2} = 2$

$\Rightarrow \lim\limits_{x\to 0}\dfrac{x^2\left(\dfrac{(\alpha-\gamma)}{x}+(\alpha+\gamma)+\dfrac{x}{3!}(\alpha-\gamma)+.....\right)}{x^2} = 2$

Since R.H.S is finite,

$\therefore \qquad \alpha - \gamma = 0 \qquad \therefore\ \alpha = \gamma, \qquad$ then $\dfrac{0+2\alpha+0+....}{1} = 2 \quad \therefore \qquad \alpha = 1$ then $\gamma = 1$

From (2), $\qquad \beta = \alpha + \gamma = 1 + 1 = 2$

So, $\alpha + 2\beta + 3\gamma = 8$

Limit when $x \to \infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x \to \infty$, we use to substitute $y = \dfrac{1}{x}$ and in this case $y \to 0^+$.

Example # 16 : Evaluate : $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}$

Solution : $\lim\limits_{x\to\infty}\dfrac{\cos x}{x} = 0$

Example # 17: Evaluate $\lim\limits_{x\to\infty} x.\tan\dfrac{1}{x}$

Solution : $\lim\limits_{x\to\infty} x.\tan\dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\tan\frac{1}{x}}{\frac{1}{x}} = 1$

Example # 18 : Evaluate : $\lim\limits_{x\to\infty} \dfrac{4x+3}{x-8}$

Solution : $\lim\limits_{x\to\infty} \dfrac{4x+3}{x-8} = \lim\limits_{x\to\infty} \dfrac{4+\frac{3}{x}}{1-\frac{8}{x}} = 4.$

Example # 19 : Evaluate $\lim\limits_{x\to\infty} \dfrac{4x^2-8}{7x+x^5+1}$

Solution : $\lim\limits_{x\to\infty} \dfrac{4x^2-8}{7x+x^5+1} = \lim\limits_{x\to\infty} \dfrac{\frac{4}{x^3}-\frac{8}{x^5}}{\frac{7}{x^4}+1+\frac{1}{x^5}} = 0$

Example # 20 : Evaluate $\lim\limits_{x\to-\infty} \dfrac{x-8}{\sqrt{4x^2+x+1}}$

Solution : Replace x by –t

$$\lim_{t\to\infty} \frac{-t-8}{\sqrt{4t^2-t+1}} = \lim_{t\to\infty} \frac{-1-\frac{8}{t}}{\sqrt{4-\frac{1}{t}+\frac{1}{t^2}}} = -\frac{1}{2}$$

Some important notes :

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$ (iii) $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$

(iv) $\lim\limits_{x\to\infty} \dfrac{(\ell nx)^n}{x} = 0$ (v) $\lim\limits_{x\to 0^+} x(\ell nx)^n = 0$

As $x \to \infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ and $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h \to 0^+$.

Example # 21 : Evaluate $\lim\limits_{x\to\infty} \dfrac{x^{10}+7x^2+1}{e^x}$

Solution : $\lim\limits_{x\to\infty} \dfrac{x^{10}+7x^2+1}{e^x} = 0$

Limits of form 1^∞, 0^0, ∞^0

(A) All these forms can be converted into $\dfrac{0}{0}$ form in the following ways

(a) If $x \to 1$, $y \to \infty$, then $z = (x)^y$ is of 1^∞ form

$\Rightarrow$ $\ell n\, z = y\, \ell n\, x$

$\Rightarrow$ $\ell n\, z = \dfrac{\ell nx}{\frac{1}{y}}$ $\left(\dfrac{0}{0} \text{ form}\right)$ As $y \to \infty \Rightarrow \dfrac{1}{y} \to 0$ and $x \to 1 \Rightarrow \ell nx \to 0$

(b) If $x \to 0$, $y \to 0$, then $z = x^y$ is of (0^0) form

$$\Rightarrow \quad \ell n\, z = y\, \ell n\, x \quad \Rightarrow \quad \ell n\ z = \frac{y}{\frac{1}{\ell n x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

(c) If $x \to \infty$, $y \to 0$, then $z = x^y$ is of $(\infty)^0$ form

$$\Rightarrow \quad \ell n\, z = y\, \ell n\, x \quad \Rightarrow \quad \ell n\ z = \frac{y}{\frac{1}{\ell n x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

(B) $(1)^\infty$ type of problems can be solved by the following method

(a) $\displaystyle \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\displaystyle \lim_{x\to a} [f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \lim_{x\to a} \left[1+f(x)-1\right]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)} = \lim_{x\to a} \left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)\ g(x)} = e^{\lim\limits_{x\to a}[f(x)-1]\ g(x)}$$

Example # 22 : Evaluate $\displaystyle \lim_{x\to\infty} \left(\frac{x^2-1}{x^2+3}\right)^{\left(x^2+1\right)}$

Solution : Since it is in the form of 1^∞

$$\left(\frac{x^2-1}{x^2+3}\right)^{\left(x^2+1\right)} = \exp\left(\lim_{x\to\infty}\left(\frac{x^2-1-x^2-3}{x^2+3}\right)\left(x^2+1\right)\right) = e^{-4}$$

Example # 23 : Evaluate $\displaystyle \lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution : Since it is in the form of 1^∞ so $\displaystyle \lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x-1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$

$$= e^{2\times\frac{\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

Example # 24 : Evaluate $\displaystyle \lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

Solution : $\displaystyle \lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put $x = a + h$

$$= \lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)} = \lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)} = e^{\lim\limits_{h\to 0}-\cot\frac{\pi h}{2a}.\left(1+\frac{h}{a+h}-1\right)}$$

$$= e^{\lim\limits_{h\to 0}-\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right).\frac{\frac{2a}{\pi}}{a+h}} = e^{-\frac{2}{\pi}}$$

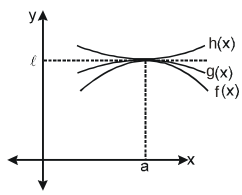
Example # 25 : Evaluate $\displaystyle \lim_{x\to 0^+} (\tan x)^{\tan x}$

Solution : Let $y = \displaystyle \lim_{x\to 0^+} (\tan x)^{\tan x}$

$$\Rightarrow \ell n\, y = \lim_{x\to 0^+} \tan x\, \ell n \tan x = \lim_{x\to 0^+} -\frac{\ell n\frac{1}{\tan x}}{\frac{1}{\tan x}} = 0, \quad \text{as } \frac{1}{\tan x} \to \infty \Rightarrow y = 1$$

Sandwitch theorem or squeeze play theorem:

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that



$\lim\limits_{x \to a} f(x) = \ell = \lim\limits_{x \to a} h(x)$,

Then $\lim\limits_{x \to a} g(x) = \ell$

Example # 26 : Evaluate $\lim\limits_{n \to \infty} \dfrac{[x] + [4x] + [7x] + ... + [(3n-2)x]}{n^2}$, where [.] denotes greatest integer function.

Solution : We know that, $x - 1 < [x] \le x$

$4x - 1 < [4x] \le 4x$

$7x - 1 < [7x] \le 7x$

. . .

. . .

$(3n - 2)x - 1 < [(3n - 2)x] \le (3n - 2)x$

$\therefore \quad (x + 4x + 7x + ... + (3n - 2)x) - n < [x] + [4x] + ...$
$+[(3n - 2)x] \le (x + 4x + ... + nx)$

$\Rightarrow \quad \dfrac{n}{2}(3n - 1)x - n < \sum\limits_{r=1}^{n} [(3r-2)x] \le \dfrac{n}{2}(3n - 1)x$

$\Rightarrow \quad \lim\limits_{n \to \infty} \dfrac{n}{2} \dfrac{(3n-1)}{n^2} x - \dfrac{1}{n} < \lim\limits_{n \to \infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} \le \lim\limits_{n \to \infty} \dfrac{n}{2} \dfrac{(3n-1)}{n^2} x$

$\Rightarrow \quad \dfrac{3x}{2} < \lim\limits_{n \to \infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} \le \dfrac{3x}{2}$

$\therefore \quad \lim\limits_{n \to \infty} \dfrac{[x]+[4x]+....+[(3n-2)x]}{n^2} = \dfrac{3x}{2}$

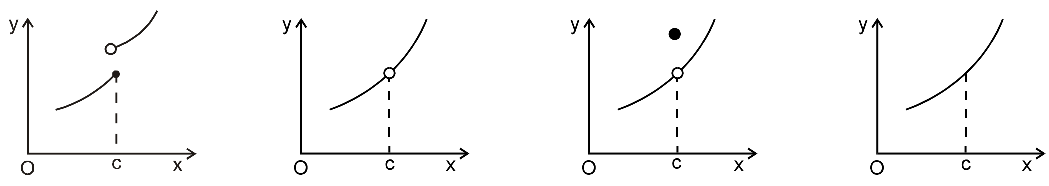
Continuity & Derivability :

A function f(x) is said to be continuous at x = c, if $\text{Limit}_{x \to c} f(x) = f(c)$

i.e. f is continuous at x = c

if $\text{Limit}_{h \to 0^+} f(c - h) = \text{Limit}_{h \to 0^+} f(c+h) = f(c)$.

If a function f (x) is continuous at x = c, the graph of f (x) at the corresponding point (c, f (c)) will not be broken. But if f (x) is discontinuous at x = c, the graph will be broken when x = c



(i) **(ii)** **(iii)** **(iv)**

((i), (ii) and (iii) are discontinuous at x = c)

((iv) is continuous at x = c)

A function f can be discontinuous due to any of the following three reasons :

(i) $\text{Limit}_{x \to c} f(x)$ does not exist i.e. $\text{Limit}_{x \to c^-} f(x) \ne \text{Limit}_{x \to c^+} f(x)$ **[figure (i)]**

(ii) f (x) is not defined at x = c **[figure (ii)]**

(iii) $\text{Limit}_{x \to c} f(x) \ne f(c)$ **[figure (iii)]**

Geometrically, the graph of the function will exhibit a break at x= c.

Example # 27 : If f(x) = $\begin{cases} \cos 2\pi x & , \quad x < 1 \\ [x] & , \quad x \ge 1 \end{cases}$, then find whether f(x) is continuous or not at x = 1, where [.] is greatest integer function.

Solution : $f(x) = \begin{cases} \cos 2\pi x & , \quad x < 1 \\ [x] & , \quad x \geq 1 \end{cases}$

For continuity at x = 1, we determine f(1) , $\lim_{x\to1^-} f(x)$ and $\lim_{x\to1^+} f(x)$.

Now, f(1) = [1] = 1

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \cos\ 2\pi x = \cos 2\pi = 1$

and $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} [x] = 1$

so $f(1) = \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1

Self practice problems :

(12) If possible find value of λ for which f(x) is continuous at $x = \frac{\pi}{2}$

$$f(x) = \begin{cases} \dfrac{1-\sin x}{1+\cos 2x}, & x < \dfrac{\pi}{2} \\ \lambda & , \quad x = \dfrac{\pi}{2} \\ \dfrac{(2x-\pi)^2}{\tan 2x} & , \quad x > \dfrac{\pi}{2} \end{cases}$$

(13) Find the values of p and q such that the function

$$f(x) = \begin{cases} x + p\sin x & ; \quad 0 \leq 4x < \pi \\ 2x\cot x + q & ; \quad \pi \leq 4x \leq 2\pi \\ \dfrac{p}{\sqrt{2}}\cos 2x - q\sin x & ; \quad 2\pi < 4x \leq 4\pi \end{cases}$$ is continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

Ans. (12) discontinuous (13) $p = \frac{\pi}{3\sqrt{2}}$, $q = \frac{-\pi}{12}$

Theorems on continuity :

(i) If f & g are two functions which are continuous at x = c, then the functions defined by:

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\,f(x)$, K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(ii) If f(x) is continuous & g(x) is discontinuous at x = a, then the product function $\phi(x) = f(x).\,g(x)$ may or may not be continuous but sum or difference function $\phi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at x = a.

e.g. f (x) = x & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(iii) If f (x) and g(x) both are discontinuous at x = a, then the product function $\phi(x) = f(x).\ g(x)$ is not necessarily be discontinuous at x = a.

e.g. $f(x) = g(x) = \begin{cases} 1 & , \quad x \geq 0 \\ -1 & , \quad x < 0 \end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

Example # 28 : If f(x) = [sin(x–1)] – {sin(x–1)}. Comment on continuity of f(x) at $x = \frac{\pi}{2} + 1$

(where [.] denotes G.I.F. and { . } denotes fractional part function).

Solution : f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [sin (x – 1)] + {sin (x – 1)} = sin (x – 1)

which is continuous at x = $\frac{\pi}{2}+1$

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at x = $\frac{\pi}{2}$ + 1

∴ At most one of f(x) or g(x) can be continuous at x = $\frac{\pi}{2}$ + 1

As g(x) is continuous at x = $\frac{\pi}{2}$ + 1, therefore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluating $\lim\limits_{x\to\frac{\pi}{2}+1}$ f(x) and f. $\left(\frac{\pi}{2}+1\right)$

Continuity of composite functions :

If f is continuous at x = c and g is continuous at x = f(c), then the composite g[f(x)] is continuous at x = c. eg. f(x) = $\frac{x\sin x}{x^2+2}$ & g(x) = $|x|$ are continuous at x = 0, hence the composite function (gof) (x) = $\left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

Self practice problem :

(14) $f(x)=\begin{cases}1+8x^3 & , \ x<0\\ -1 & , \ x=0\\ 4x^2-1 & , \ x>0\end{cases}$ and $g(x)=\begin{cases}(2x-1)^{\frac{1}{3}} & , \ x<0\\ 1 & , \ x=0\\ \sqrt{2x+1} & , \ x>0\end{cases}$

Then define fog (x) and comment on the continuity of gof(x) at x = 1/2

Ans. [$fog(x)=\begin{cases}16x-7 & ; \ x<0\\ 3 & ; \ x=0\\ 8x+3 & ; \ x>0\end{cases}$ and gof(x) is discontinous at x = 1/2]

Continuity in an Interval :

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point ∈(a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if:

(i) f is continuous in the open interval (a, b),

(ii) f is right continuous at 'a' i.e. $\underset{x\to a^+}{\text{Limit}}$ f(x) = f(a) = a finite quantity and

(iii) f is left continuous at 'b' i.e. $\underset{x\to b^-}{\text{Limit}}$ f(x) = f(b) = a finite quantity.

(c) All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

On the basis of above facts continuity of a function should be checked at the following points

(i) Continuity of a function should be checked at the points where definition of a function changes.

(ii) Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

(iii) Continuity of sgn (f(x)) should be checked at the points where f(x) = 0 (if f(x) = 0 in any open interval containing a, then x = a is not a point of discontinuity)

(iv) In case of composite function f(g(x)) continuity should be checked at all possible points of discontinuity of g(x) and at the points where g(x) = c, where x = c is a possible point of discontinuity of f(x).

Example # 29: If $f(x)=\begin{cases}[2x] & 0\le x<1\\ \{3x\}\,\text{sgn}(-x) & 1\le x\le 2\end{cases}$, where { . } represents fractional part function and [.] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

Solution : The given function is

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x < 1 \\ 3(1-x) & 1 \le x < \frac{4}{3} \\ 4-3x & \frac{4}{3} \le x < \frac{5}{3} \\ 5-3x & \frac{5}{3} \le x < 2 \\ 0 & x = 2 \end{cases}$$

so discontinous at x = 1/2 , 1,4/3, 5/3, 2

Example # 30 : If f(x) = $\frac{x+3}{x-1}$ and g(x) = $\frac{1}{x-3}$, then discuss the continuity of f(x), g(x) and fog (x).

Solution : f(x) = $\frac{x+3}{x-1}$

f(x) is a rational function it must be continuous in its domain
and f is not defined at x = 1
$\therefore$ f is discontinuous at x = 1

g(x) = $\frac{1}{x-3}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 3
$\therefore$ g is discontinuous at x = 3
Now fog (x) will be discontinuous at
(i) x = 3 (point of discontinuity of g(x))
(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x))

if g(x) = 1 $\Rightarrow \frac{1}{x-3} = 1 \quad \Rightarrow \quad x = 4$

$\therefore$ discontinuity of fog(x) should be checked at x = 3 and x = 4
at x = 3

fog (x) = $\dfrac{\frac{1}{x-3}+1}{\frac{1}{x-3}-1}$

fog (3) is not defined

$\lim\limits_{x\to 3}$ fog (x) = $\lim\limits_{x\to 3} \dfrac{\frac{1}{x-3}+1}{\frac{1}{x-3}-1} = \lim\limits_{x\to 2} \dfrac{1+x-3}{1-x+3}$ = 1 $\therefore$ fog (x) is discontinuous at x = 3

fog (4) = not defined

$\lim\limits_{x\to 4^+}$ fog (x) = ∞

$\lim\limits_{x\to 4^-}$ fog (x) = $-\infty$ $\therefore$ fog (x) i s discontinuous at x = 4.

Self practice problem :

(15) If f(x) = $\begin{cases} [\ell n\ x]\ .\ \text{sgn}\left(\left\{x-\frac{1}{2}\right\}\right) ; & 1 < x \le 3 \\ \{x^2\} ; & 3 < x \le 3.5 \end{cases}$. Find the pointswhere the continuity of f(x), should be checked, where [.] is greatest integer function and {.} fractional part function.

Ans. $\{ 1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5 \}$

Intermediate value theorem :

A function f which is continuous in $[a,b]$ possesses the following properties:

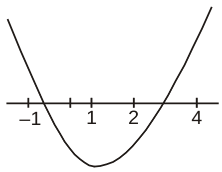
(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

Example # 31: Prove that the equation $3(x-1)(x-2) + 4(x+1)(x-4) = 0$ will have real and distinct roots.

Solution : $3(x-1)(x-2) + 4(x+1)(x-4) = 0$

$f(x) = 3(x-1)(x-2) + 4(x+1)(x-4)$
$f(-1) = +ve$
$f(1) = -ve$
$f(2) = -ve$



$f(4) = +ve$
hence $3(x-1)(x-2) + 4(x+1)(x-4) = 0$
have real and distinct roots

Self practice problem :

(16) If $f(x) = x\ell nx - 2$, then show that f(x) = 0 has exactly one root in the interval 1, e).

Example # 32: Let $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{1+n\sin^2 x}$, then find $f\left(\dfrac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Solution : Let $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n\,.\,\sin^2\frac{\pi}{4}} = \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$$

Now

$$f(0) = \lim_{n\to\infty} \frac{1}{n\,.\,\sin^2(0)+1} = \frac{1}{1+0} = 1 \Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty}\frac{1}{1+n\ \sin^2 x}\right] = 0$$

{here $\sin^2x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞}

$\therefore$ f(x) is not continuous at x = 0

Self practice problem :

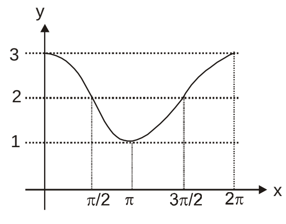
(17) If $f(x) = \lim\limits_{n\to\infty} (1+x)^n$. Comment on the continuity of f(x) at x = 0 and explain $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

Ans. Discontinous (non-removable)

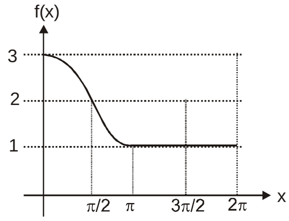
Example # 33: f(x) = minimum $(2 + \cos t,\ 0 \le t \le x),\ 0 \le x \le 2\pi$ discuss the continuity of this function at $x = \pi$

Solution : f(x) = minimum $(2 + \cos t,\ 0 \le t \le x),\ 0 \le x \le 2\pi$

$$f(x) = \begin{cases} 2+\cos x & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{cases}$$



which is continuous at $x = \pi$



Differentiability of a function at a point :

(i) The right hand derivative of f (x) at x = a denoted by $f'(a^+)$ is defined by:

R.H.D. = $f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined by:

L.H.D. = $f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a-h)-f(a)}{-h}$, provided the limit exists.

A function f(x) is said to be differentiable at x = a if $f'(a^+) = f'(a^-)$ = finite

By definition $f'(a) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$

Example # 34 : Comment on the differentiability of f(x) = $\begin{cases} 2x+3, & x<1 \\ 4x^2-1, & x\geq 1 \end{cases}$ at x = 1.

Solution : R.H.D. = $f'(1^+) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1+h)-f(1)}{h} = 8$

L.H.D. = $f'(1^-) = \underset{h\to 0^+}{\text{Limit}} \dfrac{f(1-h)-f(1)}{-h} = 2$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

Example # 35: If f(x) = $\begin{cases} ax+b, & x\leq -1 \\ ax^3+x+2b, & x>-1 \end{cases}$, then find a and b so that f(x) become differentiable at x = –1.

Solution : –a + b = –a – 1 + 2b using continuity

$\Rightarrow$ b = 1

$f'(x) = \begin{cases} a, & x<-1 \\ 3ax^2+1, & x>-1 \end{cases}$

$a = 3a + 1 \Rightarrow a = -\dfrac{1}{2}$

Example # 36 : If f(x) = $\begin{cases} [\sin\frac{3\pi}{2}x], & x\leq 1 \\ 2\{x\}-1, & x>1 \end{cases}$, then comment on the derivability at x = 1,

where [.] is greatest integer function and {.} is fractional part function.

Solution : $f'(1^-) = \lim\limits_{h\to 0^+} \dfrac{f(1-h)-f(1)}{-h} = \lim\limits_{h\to 0^+} \dfrac{\left[\sin\frac{3\pi}{2}(1-h)\right]+1}{-h} = 0$

$f'(1^+) = \lim\limits_{h\to 0^+} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{2\{1+h\}+1-1}{h} = \lim\limits_{h\to 0^+} \dfrac{2h}{h} = 2$

$\because$ $f'(1^+) \neq f'(1^-)$

f(x) is not differentiable at x = 1.

Self Practice Problems :

(18) If $f(x) = \begin{cases} \left[\frac{2x}{3}\right] + \frac{x}{3} + 2 \ , & x < 3 \\ \left\{\frac{x}{3}\right\} + 3 \ , & x \geq 3 \end{cases}$, then comment on the continuity and differentiable at $x = 3$, where [.] is greatest integer function and {.} is fractional part function.

(19) If $f(x) = \begin{cases} x\sin^{-1}1/x, & x \neq 0 \\ 0 \ , & x = 0 \end{cases}$, then comment on the derivability of f(x) at x = 0.

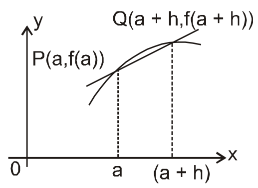
Ans. (18) Discontinuous and non-differentiable at x = 3

(19) non-differentiable at x = 0

Concept of tangent and its association with derivability :

Tangent :- The tangent is defined as the limiting case of a chord or a secant.

slope of the line joining (a,f(a)) and (a + h, f(a + h)) = $\frac{f(a+h)-f(a)}{h}$



Slope of tangent at P = $f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

The tangent to the graph of a continuous function f at the point P(a, f(a)) is

(i) the line through P with slope $f'(a)$ if $f'(a)$ exists ;

(ii) the line x = a if L.H.D. and R.H.D. both are either ∞ or $-\infty$.

If neither (i) nor (ii) holds then the graph of f does not have a tangent at the point P.

In case (i) the equation of tangent is $y - f(a) = f'(a)(x - a)$.

In case (ii) it is x = a

Note : (i) tangent is also defined as the line joining two infinitesimally close points on a curve.

(ii) A function is said to be derivable at x = a if there exist a tangent of finite slope at that point.
$f'(a^+) = f'(a^-)$ = finite value

(iii) $y = x^3$ has x-axis as tangent at origin.

(iv) y = |x| does not have tangent at x = 0 as L.H.D. $\neq$ R.H.D.

Example # 37 : Find the equation of tangent to $y = (x)^{1/3}$ at x = 1 and x = 0.

Solution : At x = 1 Here $f(x) = (x)^{1/3}$

$$\text{L.H.D} = f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+} \frac{(1-h)^{1/3}-1}{-h} = \frac{1}{3}$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^{1/3}-1}{h} = \frac{1}{3}$$

As R.H.D. = L.H.D. = $\frac{1}{3}$

$\therefore$ slope of tangent = $\frac{1}{3}$ $\therefore$ $y - f(1) = \frac{1}{3}(x-1)$

$y - 1 = \frac{1}{3}(x - 1)$

$\Rightarrow$ $3y - x = 2$ is tangent to $y = x^{1/3}$ at (1, 1)

At x = 0

L.H.D. = $f'(0^-) = \lim_{h \to 0^+} \dfrac{(0-h)^{1/3} - 0}{-h} = +\infty$

R.H.D. = $f'(0^+) = \lim_{h \to 0^+} \dfrac{(0+h)^{1/3} - 0}{h} = +\infty$

As L.H.D. and R.H.D are infinite. y = f(x) will have a vertical tangent at origin.

$\therefore$ x = 0 is the tangent to $y = x^{1/3}$ at origin.

Self Practice Problems :

(20) If possible find the equation of tangent to the following curves at the given points.
(i) $y = x^3 + 3x^2 + 28x + 1$ at x = 0.
(ii) $y = (x-8)^{2/3}$ at x = 8.
Ans. (i) y = 28x + 1 (ii) x = 8

Relation between differentiability & continuity:

(i) If $f'(a)$ exists, then f(x) is continuous at x = a.

(ii) If f(x) is differentiable at every point of its domain of definition, then it is continuous in that domain.

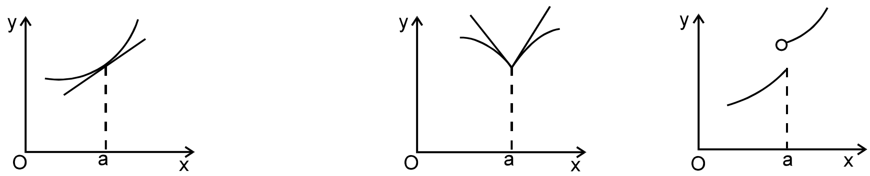
Note : The converse of the above result is not true i.e. "If 'f' is continuous at x = a, then 'f' is differentiable at x = a is not true.
e.g. the functions f(x) = | x – 2 | is continuous at x = 2 but not differentiable at x = 2.

If f(x) is a function such that R.H.D = $f'(a^+) = \ell$ and L.H.D. = $f'(a^-) = m$.

Case - I
If ℓ = m = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II
if $\ell \neq$ m = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

Case - III
If at least one of the ℓ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



(i) continuous and differentiable (ii) continuous but not differentiable (iii) neither continuous nor differentiable

Example # 38 : If f(x) is differentiable at x = a, prove that it will be continuous at x = a.

Solution : $f'(a^+) = \lim_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h} = \ell$

$\lim_{h \to 0^+}$ [f(a+h) – f(a)] = $h\ell$

as $h \to 0$ and ℓ is finite, then $\lim_{h \to 0^+}$ f(a + h) – f(a) = 0

$\Rightarrow \lim_{h \to 0^+}$ f (a + h) = f(a).

Similarly $\lim_{h \to 0^+}$ [f(a – h)– f(a)] = – $h\ell$ $\Rightarrow \lim_{h \to 0^+}$ f(a – h) = f(a)

$\therefore \lim_{h \to 0^+}$ f(a + h) = f(a) = $\lim_{h \to 0^+}$ f(a – h)

Hence, f(x) is continuous.

Example # 39 : If f(x) = $\begin{cases} \pi + x^2 \,\mathrm{sgn}[x] + \{x-4\}, & -2 \le x < 2 \\ \pi - \sin(x+\pi) + |x-3|, & 2 \le x < 6 \end{cases}$, comment on the continuity and differentiability of f(x), where [.] is greatest integer function and {.} is fractional part function, at x = 1, 2.

Solution : Continuity at x = 1

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (\pi + x^2 \operatorname{sgn}[x] + \{x - 4\}) = 1 + \pi$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (\pi + x^2 \operatorname{sgn}[x] + \{x - 4\})$

$= 1 \operatorname{sgn}(0) + 1 + \pi = 1 + \pi$

$\because \quad f(1) = 1 + \pi$

$\therefore \quad$ L.H.L = R.H.L = f(1). Hence f(x) is continuous at x = 1.

Now for differentiability,

R.H.D. $= f'(1^+) = \lim\limits_{h \to 0^+} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{\pi + (1+h)^2 \operatorname{sgn}[1+h] + \{1+h-4\} - 1 - \pi}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{(1+h)^2 + h - 1}{h} = \lim\limits_{h \to 0^+} \dfrac{1 + h^2 + 2h + h - 1}{h} = \lim\limits_{h \to 0^+} \dfrac{h^2 + 3h}{h} = 3$

and L.H.D. $= f'(1^-) = \lim\limits_{h \to 0^+} \dfrac{f(1-h) - f(1)}{-h} = \lim\limits_{h \to 0^+} \dfrac{\pi + (1-h)^2 \operatorname{sgn}[1-h] + 1 - h - 1 - \pi}{-h} = 1$

$\Rightarrow \quad f'(1^+) \neq f'(1^-)$.

Hence f(x) is non differentiable at x = 1.

Now at x = 2

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (\pi + x^2 \operatorname{sgn}[x] + \{x - 4\}) = \pi + 4 \cdot 1 + 1 = 5 + \pi$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (\pi + \sin x + |x - 3|) = 1 + \pi + \sin 2$

Hence L.H.L $\neq$ R.H.L

Hence f(x) is discontinuous at x = 2 and then f(x) also be non differentaible at x = 2.

Self Practice Problem :

(21) If $f(x) = \begin{cases} \left(\dfrac{e^{[x]} + |x| - 1}{[x] + \{2x\}}\right) & x \neq 0 \\ 1/2 & x = 0 \end{cases}$, comment on the continuity at x = 0 and differentiability at x = 0, where [.] is greatest integer function and {.} is fractional part function.

Ans. discontinuous hence non-differentiable at x = 0

Differentiability of sum, product & composition of functions :

(i) If f(x) & g(x) are differentiable at x = a, then the functions $f(x) \pm g(x)$, f(x). g(x) will also be differentiable at x = a & if g (a) $\neq$ 0, then the function f(x)/g(x) will also be differentiable at x = a.

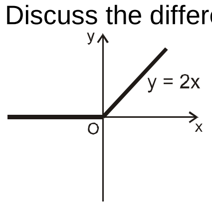
(ii) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a
e.g. f(x) = $|x|$ and g(x) = x^2.

(iii) If f(x) & g(x) both are not differentiable at x = a, then the product function
F(x) = f(x) . g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

(iv) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

Example # 40 : Discuss the differentiability of f(x) = x + |x|.

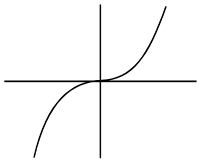
Solution :



Non-differentiable at x = 0.

Example # 41 : Discuss the differentiability of f(x) = x|x|

Solution : $\because$ $f(x) = \begin{cases} x^2 & , \quad x \ge 0 \\ -x^2 & , \quad x < 0 \end{cases}$



Differentiable at x = 0

Example # 42 : If f(x) is differentiable and g(x) is differentiable, then prove that f(x) . g(x) will be differentiable.

Solution : Given, f(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

g(x) is differentiable

i.e. $\lim\limits_{h \to 0^+} \dfrac{g(a+h) - g(a)}{h} = g'(a)$

let p(x) = f(x) . g(x)

Now, $\lim\limits_{h \to 0^+} \dfrac{p(a+h) - p(a)}{h} = \lim\limits_{h \to 0^+} \dfrac{f(a+h).g(a+h) - f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \dfrac{f(a+h)g(a+h) + f(a+h).g(a) - f(a+h).g(a) - f(a).g(a)}{h}$

$= \lim\limits_{h \to 0^+} \left[\dfrac{f(a+h)\ \ (g\ \ (a+h) - g(a))}{h} + \dfrac{g\ \ (a)(f(a+h) - f(a))}{h} \right]$

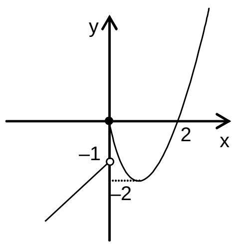
$= \lim\limits_{h \to 0^+} \left[f(a+h).\dfrac{g(a+h) - g(a)}{h} + g(a).\ \dfrac{f(a+h) - f(a)}{h} \right]$

$= f(a) . g'(a) + g(a) f'(a) = p'(a)$

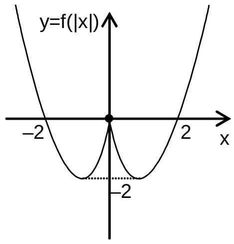
Hence p(x) is differentiable.

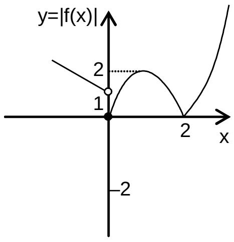
Example # 43 : If $f(x) = \begin{cases} 2x - 1 & , \quad x < 0 \\ 2x^2 - 4x & , \quad x \ge 0 \end{cases}$ then comment on the continuity and differentiability of g(x) by drawing the graph of f(|x|) and, |f(x)| and hence comment on the continuity and differentiability of g(x) = f(|x|) + |f(x)|.

Solution :



Graph of f(|x|) and |f(x)|





If f(|x|) and |f(x)| are continous, then g(x) is continuous. At x = 0 f(|x|) is continuous, and |f(x)| is discontinuous therefore g(x) is discontineous at x = 0.

$\therefore$ g(x) is non differentiable at x = 0, 2 (find the reason yourself).

Differentiability over an Interval :

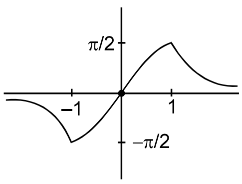
f (x) is said to be differentiable over an open interval if it is differentiable at each point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if:

(i) for the points a and b, $f'(a^+)$ and $f'(b^-)$ exist finitely

(ii) for any point c such that a < c < b, $f'(c^+)$ & $f'(c^-)$ exist finitely and are equal.

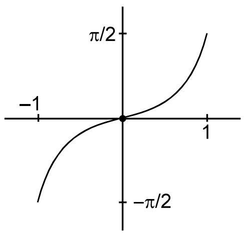
All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin^{-1} \dfrac{2x}{1+x^2}$



Non differentiable at x = 1 & x = – 1

Graph of $y = \sin^{-1} x$.



Non differentiable at x = 1 & x = – 1

Note : Derivability should be checked at following points

(i) At all points where continuity is required to be checked.

(ii) At the critical points of modulus and inverse trigonometric function.

Example # 44 : If $f(x) = \begin{cases} \left\{2x + \dfrac{7}{3}\right\}[\sin 2\pi x] & , \quad 0 \le x < \dfrac{1}{2} \\ \left([4x] + \left[\dfrac{x}{4}\right]\right).\text{sgn}\left(2x - \dfrac{4}{3}\right) & , \quad \dfrac{1}{2} \le x \le 1 \end{cases}$, find those points at which continuity and differentiability should be checked, where [.] is greatest integer function and {.} is fractional part function. Also check the continuity and differentaibility of f(x) at x = 1/2.

Solution : $f(x) = \begin{cases} \left\{2x + \dfrac{7}{3}\right\}[\sin 2\pi x] & , \quad 0 \le x < \dfrac{1}{2} \\ \left([4x] + \left[\dfrac{x}{4}\right]\right).\text{sgn}\left(2x - \dfrac{4}{3}\right) & , \quad \dfrac{1}{2} \le x \le 1 \end{cases}$

The points, where we should check the continuity and differentiability are $x = 0, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, 1$

At x = 1/2

L.H.L. = $\lim\limits_{x\to\frac{1}{2}^-} f(x) = \lim\limits_{x\to\frac{1}{2}^-} \left\{2x+\frac{7}{3}\right\}$ [sin 2πx] = 0

R.H.L. = $\lim\limits_{x\to\frac{1}{2}^+} f(x) = \lim\limits_{x\to\frac{1}{2}^+}$ [4x] sgn $\left\{2x-\frac{4}{3}\right\}$ = 2(– 1) = – 2

∴ L.H.L ≠ R.H.L. hence f(x) is discontinuous at x = 1/2 and hence it is non diffferentiable at x = 1/2.

Self Practice Problems:

(22) If f(x) = $\left[\frac{x+1}{2}\right]+\left[\frac{1-x}{2}\right]$, $-3 \le x \le 5$, then draw its graph and comment on the continuity and differentiability of f(x), where [.] is greatest integer function.

(23) If f(x) = $\begin{cases} |4x^2+6x+3| & , \quad -2 \le x < -1 \\ [x^2+2x] & , \quad -1 \le x \le 0 \end{cases}$, then draw the graph of f(x) and comment on the differentiability and continuity of f(x), where [.] is greatest integer function.

Ans. (22) f(x) is discontinuous at x = –3, –1, 1, 3, 5 hence non-differentiable.

(23) f(x) is discontinuous at x = –1, 0 & non differentiable at x = –1, 0.

Problems of finding functions satisfying given conditions :

Example # 45: If f(x) is a function satisfies the relation for all x, y ∈ R, f(x + 2y) = f(x) + f(2y) and if f′(0) = 3 and function is differentiable every where, then find f(x).

Solution : $f'(x) = \lim\limits_{h\to 0^+} \frac{f(x+2h)-f(x)}{2h} = \lim\limits_{h\to 0^+} \frac{f(x)+f(2h)-f(x)-f(0)}{2h}$ (∵ f(0) = 0)

$= \lim\limits_{h\to 0^+} \frac{f(2h)-f(0)}{2h} = f'(0) \Rightarrow f'(x) = 3 \quad \Rightarrow \quad \int f'(x)\ dx = \int 3\ dx$

f(x) = 3x + c

∵ f(0) = 2.0 + c as f(0) = 0

∴ c = 0 ∴ f(x) = 3x

Example # 46 : f(x + α) = f(x) . f(α) ∀ x, α ∈ R and f(x) is a differentiable function and f′(0) = 1/3, f(x) ≠ 0 for any x. Find f(x)

Solution : f(x) is a differentiable function

$\therefore f'(x) = \lim\limits_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0^+} \frac{f(x).f(h)-f(x).f(0)}{h}$ (∵ f(0) = 1)

$= \lim\limits_{h\to 0^+} \frac{f(x).(f(h)-f(0))}{h} = f(x).\ f'(0) = f(x) \ \therefore f'(x) = f(x) \ \therefore \int \frac{f'(x)}{f(x)}\ dx = \int \frac{1}{3} dx$

$\Rightarrow \ell n\ f(x) = \frac{x}{3} + c \ \therefore \ell n\ 1 = 0 + c \Rightarrow c = 0 \ \therefore \ell n\ f(x) = \frac{x}{3} \Rightarrow f(x) = e^{x/3}$

Example # 47 : $3f\left(\frac{x+y}{3}\right)$ = f(x) + f(y) ∀ x, y ∈ R and f(0) = 4 and f′(0) = 2 and function is differentiable for all x, then find f(x).

Solution : $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+3.0}{3}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{f(3h)-f(0)}{3h} = f'(x) = 2$

f ′(x) = 2 ⇒ f(x) = 2x + c ⇒ c = 4 ⇒ f(x) = 2x + 4

Self Practice Problem:

(24) $f\left(\frac{x}{y}\right) = f(x) - f(y)\ \forall\ x, y \in R^+$ and $f'(1) = 1$, then show that $f(x) = \ell nx$.

Result of Some Known Functional Equation :-

Let x, y are independent variables and f(x) is differentiable function in its domain :

(i) If $f(xy) = f(x) + f(y)\ \forall\ x, y \in R^+$, then $f(x) = k \ln x$ or $f(x) = 0$.

(ii) If $f(xy) = f(x).\ f(y)\ \forall\ x, y \in R$, then $f(x) = x^k$, $k \in R$

(iii) If $f(x + y) = f(x).\ f(y)\ \forall\ x, y \in R$, then $f(x) = a^{kx}$.

(iv) If $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$, then $f(x) = kx$,
where k is a constant in all four parts.

Example # 48 : If f(x) is a polynomial function satisfying $f(x)\ .\ f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(2) = 9$, then find f (3)

Solution : $f(x) = 1 \pm x^n$

As $f(2) = 9$ $\therefore$ $f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

Self practice problems

(25) If f(x) is a polynomial function satisfying $f(x)\ .\ f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(3) = -8$, then find f(4)

(26) If $f(x + y) = f(x)\ .\ f(y)$ for all real x, y and $f(0) \neq 0$, then prove that the function, $g(x) = \frac{f(x)}{1+f^2(x)}$ is an even function.

Ans. (25) – 15

Example # 49 : Evaluate $\lim\limits_{h\to 0} \frac{f(\alpha - 9h) - f(\alpha + h^2)}{h}$, if $f'(\alpha) = 2$

Solution : $\because$ $\lim\limits_{h\to 0} \left(\frac{f(\alpha - 9h) - f(\alpha + h^2)}{(-9h - h^2)h}\right).(-9h - h^2) = \lim\limits_{h\to 0} f'(\alpha)\ .\ (-9 - h) = 2 \times -9 = -18$

Self Practice Problems :

(27) If f(x) and g(x) are differentiable, then prove that $f(x) \pm g(x)$ will be differentiable.

(28) If $f'(3) = 12$, then find the value of $\lim\limits_{h\to 0} \frac{f(3 + h) - f(3 + \sinh)}{h.\tan^2 h}$.

Ans. (28) 2