Indefinite Integration

But just as much as it is easy to find the differential of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not. Bernoulli, Johann

If f & g are functions of x such that $g'(x) = f(x)$, then indefinite integration of f(x) with respect to x is defined and denoted as $\int f(x)\,dx = g(x) + C$, where C is called the **constant of integration**.

Standard Formula:

(i) $\int (ax + b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\,\ell n\,|ax + b| + C$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + C$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a} + C;\ a > 0$

(v) $\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + C$

(vi) $\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + C$

(vii) $\int \tan(ax + b)\,dx = \frac{1}{a}\,\ell n\,|\sec(ax + b)| + C$

(viii) $\int \cot(ax + b)\,dx = \frac{1}{a}\,\ell n\,|\sin(ax + b)| + C$

(ix) $\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b) + C$

(x) $\int \text{cosec}^2(ax + b)\,dx = -\frac{1}{a}\cot(ax + b) + C$

(xi) $\int \sec(ax + b).\tan(ax + b)\,dx = \frac{1}{a}\sec(ax + b) + C$

(xii) $\int \text{cosec}(ax + b).\cot(ax + b)\,dx = -\frac{1}{a}\text{cosec}(ax + b) + C$

(xiii) $\int \sec x\,dx = \ell n\,|\sec x + \tan x| + C$ **OR** $\ell n\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(xiv) $\int \text{cosec}\,x\,dx = \ell n\,|\text{cosec}x - \cot x| + C$ **OR** $\ell n\left|\tan\frac{x}{2}\right| + C$ **OR** $-\ell n\,|\text{cosec}x + \cot x| + C$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(xvii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(xviii) $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right| + C$ **OR** $\sinh^{-1}\dfrac{x}{a} + C$

(xix) $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right| + C$ **OR** $\cosh^{-1}\dfrac{x}{a} + C$

(xx) $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\ \ln\left|\dfrac{a+x}{a-x}\right| + C$

(xxi) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ \ln\left|\dfrac{x-a}{x+a}\right| + C$

(xxii) $\int \sqrt{a^2-x^2}\ dx = \dfrac{x}{2}\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C$

(xxiii) $\int \sqrt{x^2+a^2}\ dx = \dfrac{x}{2}\sqrt{x^2+a^2} + \dfrac{a^2}{2}\ \ell n\left|\dfrac{x+\sqrt{x^2+a^2}}{a}\right| + C$

(xxiv) $\int \sqrt{x^2-a^2}\ dx = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ \ell n\left|\dfrac{x+\sqrt{x^2-a^2}}{a}\right| + C$

(xxv) $\int e^{ax}.\sin bx\ dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

(xxvi) $\int e^{ax.}\cos bx\ dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

Theorems on integration

(i) $\int C\ f(x).dx = C\int f(x).dx$

(ii) $\int (f(x) \pm g(x))\ dx = \int f(x)dx \pm \int g(x)\ dx$

(iii) $\int f(x)dx = g(x) + C_1 \Rightarrow \int f(ax+b)dx = \dfrac{g(ax+b)}{a} + C_2$

Example # 1 Evaluate : $\int 3x^6 dx$

Solution : $\int 3x^6 dx = \dfrac{3}{7}x^7 + C$

Example # 2 Evaluate : $\int \left(x^3+5x^2-4+\dfrac{7}{x}+\dfrac{2}{\sqrt{x}}\right) dx$

Solution : $\int \left(x^3+5x^2-4+\dfrac{7}{x}+\dfrac{2}{\sqrt{x}}\right) dx$

$= \int x^3\ dx + \int 5x^2\ dx - \int 4dx + \int \dfrac{7}{x}dx + \int \dfrac{2}{\sqrt{x}}dx$

$= \int x^3\ dx + 5\int x^2\ dx. - 4.\int 1\ .\ dx + 7.\int \dfrac{1}{x}dx + 2.\int x^{-1/2}dx$

$= \dfrac{x^4}{4} + 5.\dfrac{x^3}{3} - 4x + 7\ \ell n\ |x| + 2\left(\dfrac{x^{1/2}}{1/2}\right) + C = \dfrac{x^4}{4} + \dfrac{5}{3}x^3 - 4x + 7\ \ell n\ |x| + 4\sqrt{x} + C$

Example # 3 Evaluate : $\int 2^{x\log_2 3}\ dx$

Solution : We have, $\int 2^{x\log_2 3}\ dx = \int 3^x dx = \dfrac{3^x}{\ell n 3} + C$

Example # 4 Evaluate : $\int \frac{4^x + 5^x}{7^x}\ dx$

Solution : $\int \frac{4^x + 5^x}{7^x}\ dx = \int \left(\frac{4^x}{7^x} + \frac{5^x}{7^x}\right) dx = \int \left[\left(\frac{4}{7}\right)^x + \left(\frac{5}{7}\right)^x\right] dx = \frac{(4/7)^x}{\ell n\left(\frac{4}{7}\right)} + \frac{(5/7)^x}{\ell n\left(\frac{5}{7}\right)} + C$

Example # 5 Evaluate : $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} dx$

Solution : We have, $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} dx = \frac{1}{2} \int \frac{2\sin\frac{5x}{2}\cos 7x - 2\sin\frac{5x}{2}\cos 8x}{\sin\frac{5x}{2} + 2\cos 5x \sin\frac{5x}{2}} dx$

$= \frac{1}{2} \int \frac{\left(\sin\frac{19x}{2} - \sin\frac{9x}{2}\right) - \left(\sin\frac{21x}{2} - \sin\frac{11x}{2}\right)}{\sin\frac{5x}{2} + \sin\frac{15x}{2} - \sin\frac{5x}{2}} dx$

$= \frac{1}{2} \int \frac{\left(\sin\frac{19x}{2} + \sin\frac{11x}{2}\right) - \left(\sin\frac{21x}{2} + \sin\frac{9x}{2}\right)}{\sin\frac{15x}{2}} dx$

$= \frac{1}{2} \int \frac{2\sin\frac{15x}{2}\cos 2x - 2\sin\frac{15x}{2}\cos 3x}{\sin\frac{15x}{2}} dx = \int \cos 2x - \cos 3x) dx = \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + C$

Example # 6 Evaluate : $\int \frac{x^3}{(x+1)^2} dx$

Solution : $\int \frac{x^3}{(x+1)^2} dx = \int \frac{x^3 + 1 - 1}{(x+1)^2} dx = \int \frac{x^3 + 1}{(x+1)^2} dx - \int \frac{1}{(x+1)^2} dx$

$= \int \frac{(x+1)(x^2 - x + 1)}{(x+1)^2} dx - \int \frac{1}{(x+1)^2} dx = \int \frac{x^2 - x + 1}{(x+1)} dx - \int \frac{1}{(x+1)^2} dx$

$= \int \left(x - 2 + \frac{3}{x+1}\right) dx - \int \frac{1}{(x+1)^2} dx = \frac{x^2}{2} - 2x + 3\ \ell n(x+1) + \frac{1}{x+1} + C$

Example # 7 : Evaluate : $\int \frac{1}{4 + 9x^2}\ dx$

Solution : We have

$\int \frac{1}{4 + 9x^2}\ dx = dx\ \frac{1}{9} \int \frac{1}{\frac{4}{9} + x^2} = \frac{1}{9} \int \frac{1}{(2/3)^2 + x^2} dx$

$= \frac{1}{9} . \frac{1}{(2/3)} \tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6} \tan^{-1}\left(\frac{3x}{2}\right) + C$

Example # 8 : Evaluate : $\int \cos x \cos 2x\ \ dx$

Solution : $\int \cos x \cos 2x\ \ dx = \frac{1}{2} \int 2\cos x\ \cos 2x\ \ dx = \frac{1}{2} \int (\cos 3x + \cos x)\ dx = \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C$

Self Practice Problems :

(1) Evaluate : $\int \tan^2 x\ dx$ (2) Evaluate : $\int \frac{1}{1 + \sin x}\ dx$

Ans. (1) $\tan x - x + C$ (2) $\tan x - \sec x + C$

Integration by Substitution

If we substitute $\phi(x) = t$ in an integral then
(i) everywhere x will be replaced in terms of new variable t.
(ii) dx also gets converted in terms of dt.

Example # 9 : Evaluate : $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\, dx$

Solution : We have,

$$\int \frac{\cos x + x\sin x}{x(x+\cos x)}\, dx$$

$$= \int \frac{(x+\cos x) - x + x\sin x}{x(x+\cos x)}\, dx = \int \left(\frac{1}{x} - \frac{1-\sin x}{x+\cos x}\right) dx = \int \frac{1}{x}\, dx - \int \frac{1-\sin x}{x+\cos x} dx$$

$$= \int \frac{1}{x}\, dx - \int \frac{1}{x+\cos x} d\,(x+\cos x) = \ell n|x| - \ell n\,|x + \cos x| + C.$$

Example # 10 : Evaluate : $\int \frac{(\ell nx)^n}{x} dx$

Solution : We have $\int \frac{(\ell nx)^n}{x} dx = \int (\ell nx)^n \frac{1}{x} dx = \int (\ell nx)^n d(\ell nx) = \frac{(\ell nx)^{n+1}}{n+1} + C$

Example # 11 : Evaluate : $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$

Solution : We have , $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx = \int (\sin^{-1} x)^3 d(\sin^{-1} x) = \frac{(\sin^{-1} x)^4}{4} + C$

Example # 12 : Evaluate : $\int \frac{x}{x^4 + 2x^2 + 2}\, dx$

Solution : We have,

$$I = \int \frac{x}{x^4 + 2x^2 + 2}\, dx = \int \frac{x}{(x^2)^2 + 2x^2 + 2}\, dx \qquad \{\text{Put } x^2 = t \Rightarrow x.dx = \frac{dt}{2}\}$$

$$\Rightarrow \quad I = \frac{1}{2} \int \frac{1}{t^2 + 2t + 2}\, dt = \frac{1}{2} \int \frac{1}{(t+1)^2 + 1}\, dt = \frac{1}{2} \tan^{-1}(t+1) + C$$

$$= \frac{1}{2} \tan^{-1}(x^2 + 1) + C$$

Note:

(i) $\int [f(x)]^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + C$

(ii) $\int \frac{f'(x)}{[f(x)]^n}\, dx = \frac{(f(x))^{1-n}}{1-n} + C,\ n \neq 1$

(iii) $\int \frac{dx}{x(x^n+1)}$; $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(iv) $\int \frac{dx}{x^2 \left(x^n+1\right)^{(n-1)/n}}$; $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(v) $\int \int \frac{dx}{x^n \left(1+x^n\right)^{1/n}}$; take x^n common as x and put $1 + x^{-n} = t$.

Self Practice Problems :

(3) Evaluate : $\int \frac{\sec^2 x}{1+\tan x} dx$ (4) Evaluate : $\int \frac{\sin(\ell n x)}{x} dx$

Ans. (3) $\ell n\ |1 + \tan x| + C$ (4) $-\cos(\ell n\ x) + C$

Integration by Parts : Product of two functions f(x) and g(x) can be integrated using formula :

$$\int (f(x)\ g(x))\ dx = f(x)\int (g(x))\,dx - \int\left(\frac{d}{dx}(f(x))\ \int (g(x))\,dx\right)dx$$

(i) when you find integral $\int g(x)\ dx$ then it will **not** contain arbitarary constant.

(ii) $\int g(x)\ dx$ should be taken as same at both places.

(iii) The choice of f(x) and g(x) can be decided by ILATE guideline.
the function will come later is taken an integral function (g(x)).

I → Inverse function
L → Logarithmic function
A → Algebraic function
T → Trigonometric function
E → Exponential function

Example # 13 : Evaluate : $\int \sec^{-1}x\ dx$

Solution : Put $\sec^{-1} x = t$ so that $x = \sec t$ and $dx = \sec t \tan t\ dt$

$\therefore \int \sec^{-1}x\ dx = \int \underset{I}{t}(\underset{II}{\sec t}\ \tan t)dt = t\,(\sec t) - \int 1.\sec t\ dt$

$= t \sec t - \ell n\ |\sec t + \tan t)| + C$

$= t \sec t - \ell n\ |\sec t + \sqrt{\sec^2 t - 1}\ | + C = x\,(\sec^{-1}x) - \ell n\ |x + \sqrt{x^2-1}\ | + C$

Example # 14 : Evaluate : $\int x\ \ell n(1+x)\ dx$

Solution : Let $I = \int x\ \ell n(1+x)\ dx = \frac{x^2}{2}.\ell n\ (x+1) - \int \frac{1}{x+1}.\frac{x^2}{2}\ dx$

$= \frac{x^2}{2}\ \ell n\ (x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\ dx = \frac{x^2}{2}\ \ell n\ (x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\ dx$

$= \frac{x^2}{2}\ \ell n\ (x+1) - \frac{1}{2}\int\left(\frac{x^2-1}{x+1} + \frac{1}{x+1}\right) dx = \frac{x^2}{2}\ \ell n\ (x+1) - \frac{1}{2}\int\left((x-1) + \frac{1}{x+1}\right) dx$

$= \frac{x^2}{2}\ \ell n\ (x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \ell n\,|\,x+1|\right] + C$

Example # 15 : Evaluate : $\int e^{2x} \sin 3x\ dx$

Solution : Let $I = \int e^{2x}\sin 3x\ dx$

$= e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\left(-\frac{\cos 3x}{3}\right)dx = -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$

$= -\frac{1}{3}\ e^{2x}\ \cos 3x + \frac{2}{3}\left[e^{2x}\frac{\sin 3x}{3} - \int 2e^{2x}\frac{\sin 3x}{3}\ dx\right]$

$= -\frac{1}{3}e^{2x}\ \cos 3x + \frac{2}{9}\ e^{2x}\ \sin 3x - \frac{4}{9}\int e^{2x}\sin 3x\ dx$

$\Rightarrow I = -\frac{1}{3}\ e^{2x}\ \cos 3x + \frac{2}{9}e^{2x}\ \sin 3x - \frac{4}{9}\ I \quad \Rightarrow \quad I + \frac{4}{9}\ I = \frac{e^{2x}}{9}\ (2 \sin 3x - 3 \cos 3x)$

$\Rightarrow \frac{13}{9}\ I = \frac{e^{2x}}{9}\ (2 \sin 3x - 3 \cos 3x) \Rightarrow I = \frac{e^{2x}}{13}\ (2 \sin 3x - 3 \cos 3x) + C$

Note :

(i) $\int e^x [f(x) + f'(x)]\, dx = e^x\, f(x) + C$ (ii) $\int [f(x) + xf'(x)]\, dx = x\, f(x) + C$

Example # 16 : Evaluate : $\int e^x \frac{(x^2-2x+2)}{(x^2+2)^2} dx$

Solution : Given integral = $\int e^x \frac{(x^2-2x+2)}{(x^2+2)^2} dx = \int e^x \left\{\frac{1}{x^2+2} + \frac{(-2x)}{(x^2+2)^2}\right\} = \frac{e^x}{x^2+2} + C$

Example # 17 : Evaluate : $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

Solution : Given integral $= \int e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$

$= \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right) dx = -e^x \cot\frac{x}{2} + C$

Example # 18 : Evaluate : $\int \left[\ell n\ (\ell nx) + \frac{1}{(\ell nx)^2}\right] dx$

Solution : Let $I = \int \left[\ell n\ (\ell nx) + \frac{1}{(\ell nx)^2}\right] dx$ {put $x = e^t \Rightarrow dx = e^t dt$}

$\therefore \quad I = \int e^t \left(\ell nt + \frac{1}{t^2}\right) dt \int e^t \left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right) = dt$

$= e^t \left(\ell nt - \frac{1}{t}\right) + C = x\left[\ell n\ (\ell nx) - \frac{1}{\ell nx}\right] + C$

Self Practice Problems :

(5) Evaluate : $\int x \sin x\ dx$ (6) Evaluate : $\int x^2 e^x\ dx$

Ans. (5) $-x\cos x + \sin x + C$ (6) $x^2 e^x - 2xe^x + 2e^x + C$

Integration of type $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

Example # 19 : Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Solution : We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\ dx = \int \sqrt{(x+1)^2+2^2}$

$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}.(2)^2 \ell n\,|(x+1) + \sqrt{(x+1)^2+2^2}| + C$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\,\ell n\,|(x+1) + \sqrt{x^2+2x+5}| + C$

Example # 20 : Evaluate : $\int \frac{dx}{\sqrt{2-6x-9x^2}}$ dx

Solution : $\int \frac{dx}{\sqrt{2-6x-9x^2}}$ dx $= \int \frac{1}{\sqrt{3-(3x+1)^2}}$ dx $= \frac{1}{3}\sin^{-1}\left(\frac{3x+1}{\sqrt{3}}\right) + C$

Self Practice Problems :

(7) Evaluate : $\int \frac{1}{2x^2+x-1}$ dx

(8) Evaluate : $\int \frac{8x-11}{\sqrt{5+2x-x^2}}$ dx

Ans. (7) $\frac{1}{3}\ \ell n \left|\frac{2x-1}{2x+2}\right| + C$ (8) $-8\sqrt{5+2x-x^2} - 3\sin^{-1}\frac{x-1}{\sqrt{6}} + C$

Integration of type

$$\int \frac{px+q}{ax^2+bx+c}dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,\ \int (px+q)\sqrt{ax^2+bx+c}\ dx$$

Express px + q = A (differential co–efficient of denominator) + B.

Example # 21 : Evaluate : $\int \frac{2x-3}{x^2+3x-18}dx$

Solution : Let $2x - 3 = \lambda \frac{d}{dx}(x^2 + 3x - 18) + \mu$

Then $2x - 3 = \lambda(2x + 3) + \mu$

Comparing the coefficients of like power of x, we get.

$2\lambda = 2$, and $3\lambda + \mu = -3$ $\Rightarrow \lambda = 1$ and $\mu = -6$

So, $\int \frac{2x-3}{x^2+3x-18}dx = \int \frac{2x+3-6}{x^2+3x-18}dx = \int \frac{2x+3}{x^2+3x-18}dx - 6\int \frac{1}{x^2+3x-18}dx$

$= \ell n|x^2 + 3x - 18| - 6\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}-18}dx = \ell n|x^2 + 3x - 18| - 6\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx$

$= \ell n|x^2 + 3x - 18| - 6.\frac{1}{2\left(\frac{9}{2}\right)}\ \ell n\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C = \ell n|x^2 + 3x - 18| - \frac{2}{3}\ \ell n\left|\frac{x-3}{x+6}\right| + C$

Example # 22 : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}$ dx

Solution : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}$ dx $= \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}$ dx $= \int \frac{2x+4}{\sqrt{x^2+4x+1}}$ dx $- \int \frac{1}{\sqrt{x^2+4x+1}}$ dx

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}$ dx, where $t = (x^2 + 4x + 1)$ for I[st] integral

$= 2\sqrt{t} - \ell n\,|(x + 2) + \sqrt{x^2+4x+1}| + C = 2\sqrt{x^2+4x+1} - \ell n\,|x + 2 + \sqrt{x^2+4x+1}| + C$

Example # 23 : Evaluate : $\int x\sqrt{1+x-x^2}dx$

Solution : Let $x = \lambda \,.\, \frac{d}{dx}(1+x-x^2) + \mu$.

$\Rightarrow x = \lambda(1-2x) + \mu$

Comparing the coefficients of like powers of x, we get

$1 = -2\lambda$ and $\lambda + \mu = 0 \Rightarrow \lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}$ $\therefore$ $x = -\frac{1}{2}(1-2x) + \frac{1}{2}$

so, $\int x\sqrt{1+x-x^2}\,dx$

$= \int\left\{-\frac{1}{2}(1-2x)+\frac{1}{2}\right\}\sqrt{1+x-x^2}dx = -\frac{1}{2}\int(1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int\sqrt{1+x-x^2}\,dx$

$= -\frac{1}{2}\int\sqrt{1+x-x^2}\,d(1+x-x^2) + \frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx,$

$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left[\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2\sin^{-1}\frac{x-1/2}{\sqrt{5}/2}\right] + C$

$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left[\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right] + C$

Self Practice Problems :

(9) Evaluate : $\int\frac{3-4x}{2x^2-3x+1}\,dx$

(10) Evaluate : $\int\frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$

(11) Evaluate : $\int(x-1)\sqrt{1+x+x^2}\,dx$

Ans. (9) $-\ell n|2x^2-3x+1| + C$

(10) $2\sqrt{3x^2-5x+1} + C$

(11) $\frac{1}{3}(x^2+x+1)^{3/2} - \frac{3}{8}(2x+1)\sqrt{1+x+x^2} - \frac{9}{16}\log\left(2x+1+2\sqrt{x^2+x+1}\right) + C$

Integration of Rational Algebraic Functions by using Partial Fractions:

PARTIAL FRACTIONS :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) ≥ degree of g(x) then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide f(x) by g(x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\Psi(x)}{g(x)}$, where $\phi(x)$ and $\Psi(x)$ are polynomials such that the degree of $\Psi(x)$ is less than that of g(x). Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

CASE-I $\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{x-\gamma}$

CASE-II $\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{(x-\beta)^2}$

CASE-III $\frac{ax^2+bx+c}{(x-\alpha)(x^2+\beta^2)} = \frac{A}{x-\alpha}+\frac{Bx+C}{x^2+\beta^2}$

where A, B, C can be evaluated by subsitution or by comparing coefficients.

Example # 24 : Resolve $\frac{1}{2x^3 + 3x^2 - 3x - 2}$ into partial fractions.

Solution : We have, $\frac{1}{2x^3 + 3x^2 - 3x - 2} = \frac{1}{(x-1)(x+2)(2x+1)}$

Let $\frac{1}{2x^3 + 3x^2 - 3x - 2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{2x+1}$. Then,

$\Rightarrow$ 1 = A(x + 2)(2x + 1) + B(x – 1) (2x + 1) + C (x – 1)(x + 2) ..(i)

Putting x – 1 = 0 or x = 1 in (i), we get $\Rightarrow$ $A = \frac{1}{9}$,

Putting x = –2 in (i), we obtain $B = \frac{1}{9}$

Putting $x = -\frac{1}{2}$ in (i), we obtain $C = -\frac{4}{9}$

$\therefore$ $\frac{1}{2x^3 + 3x^2 - 3x - 2} = \frac{1}{(x-1)(x+2)(2x+1)} = \frac{1}{9(x-1)} + \frac{1}{9(x+2)} - \frac{4}{9(2x+1)}$

Example # 25 : Resolve $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$ into partial fractions.

Solution : Here the given function is an improper rational function. On dividing we get

$\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = x - 1 + \frac{(-x+4)}{(x^2 - 5x + 6)}$ ....(i)

we have, $\frac{-x+4}{x^2 - 5x + 6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$, then

– x + 4 = A(x – 3) + B(x – 2) ....(ii)

Putting x – 3 = 0 or x = 3 in (ii), we get

1 = B(1) $\Rightarrow$ B = 1.

Putting x – 2 = 0 or x = 2 in (ii), we get

2 = A (2 – 3) $\Rightarrow$ A = – 2

$\therefore \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2} + \frac{1}{x-3}$

Hence $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = x - 1 - \frac{2}{x-2} + \frac{1}{x-3}$

Example # 26 : Evaluate : $\int \frac{3x+1}{(x-1)^3(x+1)} dx$

Solution : Let $\frac{3x+1}{(x-1)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$ .... (i)

Multiplying both sides by (x + 1) and then putting x = – 1, we get

$A = \frac{-2}{(-2)^3} = \frac{1}{4}$

Multiplying both sides by $(x - 1)^3$ and then putting x = 1, we get

$D = \frac{4}{2} = 2$

From (i) , we get

$3x + 1 = A (x - 1)^3 + B (x - 1)^2 (x + 1) + C (x - 1) (x + 1) + D (x + 1)$

putting x = 0, we get

1 = –A + B – C + D

$\Rightarrow \quad 1 = -\frac{1}{4} + B - C + 2 \quad \Rightarrow B - C = \frac{-3}{4}$

Putting x = 2, we get

$7 = A + 3B + 3C + 3D$

$\Rightarrow \quad 7 = \frac{1}{4} + 3B + 3C + 6 \quad \Rightarrow 3B + 3C = \frac{3}{4} \Rightarrow B + C = \frac{1}{4}$

Solving $B + C = \frac{1}{4}$ and $B - C = \frac{-3}{4}$, we get $B = -\frac{1}{4}$, $C = \frac{1}{2}$

Substituting the values of A, B, C and D in (i), we get

$$\Rightarrow \quad \frac{3x+1}{(x-1)^3(x+1)} = \frac{1}{4}.\frac{1}{x+1} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{2}{(x-1)^3}$$

$$\Rightarrow \quad \int \frac{3x+1}{(x-1)^3(x+1)}dx = \frac{1}{4}\int \frac{1}{x+1}dx - \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{(x-1)^2}dx + 2\int \frac{1}{(x-1)^3}dx$$

$$= \frac{1}{4}\ell n\,|x+1| - \frac{1}{4}\ell n|x-1| - \frac{1}{2(x-1)} - \frac{1}{(x-1)^2} + C$$

Example # 27 : Evaluate : $\int \frac{1}{\sin x(2\cos^2 x - 1)}dx$

Solution : Putting cosx = t, we get

$$I = \int \frac{1}{\sin x(2\cos^2 x - 1)}dx = \int \frac{1}{\sin x(2t^2 - 1)} \times -\frac{dt}{\sin x} = -\int \frac{1}{(1-t^2)(2t^2-1)}dt$$

$$\therefore \quad I = -\int\left(\frac{1}{1-t^2} + \frac{2}{2t^2-1}\right)dt = -\int \frac{1}{1-t^2}dt - 2\int \frac{1}{2t^2-1}dt$$

$$= -\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right| - \frac{\sqrt{2}}{2}\ell n\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C = -\frac{1}{2}\ell n\left|\frac{1+\cos x}{1-\cos x}\right| - \frac{1}{\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C$$

Example # 28 : Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution : Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting x = 1 in (i), we get $-1 = A(1 + 1)^2 \Rightarrow A = -$

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0$, $C - B = 0$, $2A + B - C + D = 0$, $C + E - B - D = 2$ and $A - C - E = -3$.

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$B = \frac{1}{4} = C$, $D = \frac{1}{4}$ and $E = \frac{5}{2}$ $\quad \therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$

Example # 29 : Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Solution : We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

Then, $2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ...(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting x = 0 in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting x = – 1 in (i), we get –2 = A + 2B – 2 C. $\Rightarrow$ $-2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$\therefore$ $\frac{2x}{x^3-1} = \frac{2}{3} \cdot \frac{1}{x-1} + \frac{(-\;2/3)\;\; x+2/3}{x^2+x+1}$ or $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$

Self Practice Problems :

(12) (i) Evaluate : $\int \frac{1}{(x+2)(x+3)}dx$ (ii) Evaluate : $\int \frac{dx}{(x+1)(x^2+1)}$

Ans. (12) (i) $\ell n \left|\frac{x+2}{x+3}\right| + C$ (ii) $\frac{1}{2}\ell n\,|x + 1| - \ell n\,(x^2 + 1) + \frac{1}{2}\tan^{-1}(x) + C$

Integration of type

$\int \frac{x^2 \pm 1}{x^4 + Kx^2 + 1}$ **dx where K is any constant.**

Divide Nr & Dr by x² & put $x \mp \frac{1}{x} = t$.

Example # 30 : Evaluate $\int \frac{x^2+4}{x^4+16}dx$

Solution : $\int \frac{x^2+4}{x^4+16}dx = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx = \int \frac{1}{\left(x-\frac{4}{x}\right)^2+8}d\left(x-\frac{4}{x}\right) = \int \frac{dt}{t^2+(2\sqrt{2})^2}$,

where $t = x - \frac{4}{x}$ $= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C$

Example # 31 : Evaluate : $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx$

Solution : $\Rightarrow I = \int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}dx$ $\left[\begin{matrix}\text{Multiplying the} \\ N^r \text{ and } D^r \text{ by } (x+1)\end{matrix}\right]$

$\Rightarrow I = \int \frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}dx$

$\Rightarrow I = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$ [Dividing N^r and D^r by x^2)

$\Rightarrow I = \int \frac{2t\;\;dt}{(t^2+1)\sqrt{t^2}}$ where, $x + \frac{1}{x} + 1 = t^2$ $\Rightarrow I = 2\int \frac{1}{t^2+1}dt$ $\Rightarrow I = 2\tan^{-1}(t) + C$

$\Rightarrow I = 2\tan^{-1}\sqrt{x+\frac{1}{x}+1} + C$

Self Practice Problems :

(13) Evaluate : $\int \frac{x^2-1}{x^4-7x^2+1}dx$ (14) Evaluate : $\int \sqrt{\tan x}\;dx$

Ans. (13) $\frac{1}{6}\ell n\left|\frac{x+\frac{1}{x}-3}{x+\frac{1}{x}+3}\right| + C$ (14) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ell n\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right| + C$

where $y = \sqrt{\tan x} - \frac{1}{\sqrt{\tan x}}$

Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \textbf{ OR } \int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}}.$$

Put px + q = t^2.

Example # 32 : Evaluate : $\int \frac{dx}{(x-4)\sqrt{x+5}}$

Solution : Let I = $\int \frac{dx}{(x-4)\sqrt{x+5}}$ {Put x + 5 = t^2 $\Rightarrow$ dx = 2t dt}

$\therefore$ I = $\int \frac{2dt}{\left(t^2-9\right)} = \frac{2}{6}\ \ell n \left|\frac{t-3}{t+3}\right| + C = \frac{1}{3}\ \ln \left|\frac{\sqrt{x+5}-3}{\sqrt{x+5}+3}\right| + C$

Example # 33 : Evaluate : $\int \frac{dx}{\left(x^2+3x+2\right)\sqrt{x+4}}$

Solution : Let I = $\int \frac{dx}{\left(x^2+3x+2\right)\sqrt{x+4}}$

Putting x + 4 = t^2, and dx = 2t dt, we get I = $\int \frac{2t\ \ dt}{\{(t^2-4)^2+3(t^2-4)+2\}\sqrt{t^2}}$

$\Rightarrow$ $2\int \frac{dt}{t^4-5t^2+6}$ dt = $2\int \frac{dt}{\left(t^2-2\right)\left(t^2-3\right)}$ dt = $2\int\left[\frac{1}{t^2-3}-\frac{1}{t^2-2}\right]dt$

$= \frac{1}{\sqrt{3}}\ \ell n \left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \frac{1}{\sqrt{2}}\ \ell n \left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$ where t = $\sqrt{x+4}$

Integration of type

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, **put ax + b = $\frac{1}{t}$** ; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, **put x = $\frac{1}{t}$**

Example # 34 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$

Solution : Let I = $\int \frac{dx}{(x-1)\sqrt{x^2-x-1}}$ {put x – 1 = $\frac{1}{t}$ $\Rightarrow$ dx = $-\frac{1}{t^2}$ dt }

$$\Rightarrow \quad I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2-\left(\frac{1}{t}+1\right)-1}} = \int -\frac{dt}{\sqrt{-t^2+t+1}} = \int -\frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}$$

$$= -\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C = -\sin^{-1}\left(\frac{2t-1}{\sqrt{5}}\right) + C \text{ , where } t = \frac{1}{x-1}$$

Example # 35 : Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Put $x = \frac{1}{t} \Rightarrow \quad dx = -\frac{1}{t^2}\,dt \quad \Rightarrow \quad I = -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$ {put $t^2 - 1 = y^2 \Rightarrow t\,dt = y\,dy$ }

$\Rightarrow \quad I = -\int \frac{y\ dy}{(y^2+2)\ y} = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$

$= -\frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

Self Practice Problems :

(15) Evaluate : $\int \frac{dx}{(x+2)\sqrt{x+1}}$

(16) Evaluate : $\int \frac{dx}{(x^2+5x+6)\sqrt{x+1}}$

(17) Evaluate : $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$

(18) Evaluate : $\int \frac{dx}{(2x^2+1)\sqrt{1-x^2}}$

(19) Evaluate : $\int \frac{dx}{(x^2+2x+2)\sqrt{x^2+2x-4}}$

Ans. (15) $2\tan^{-1}\left(\sqrt{x+1}\right) + C$ (16) $2\tan^{-1}\left(\sqrt{x+1}\right) - \sqrt{2}\ \tan^{-1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right) + C$

(17) $\sin^{-1}\left(\frac{\frac{3}{2}-\frac{1}{x+1}}{\frac{\sqrt{5}}{2}}\right) + C$ (18) $-\frac{1}{\sqrt{3}}\ \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{3}\ x}\right) + C$

(19) $-\frac{1}{2\sqrt{6}}\ \ell n\left(\frac{\sqrt{x^2+2x-4}-\sqrt{6}\ (x+1)}{\sqrt{x^2+2x-4}+\sqrt{6}\ (x+1)}\right) + C$

Integration of type

$\int \sqrt{\frac{x-\alpha}{\beta-x}}$ **dx or** $\int \sqrt{(x-\alpha)(\beta-x)}$ **dx; put** $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}$ **dx or** $\int \sqrt{(x-\alpha)(x-\beta)}$ **dx; put** $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ **; put** $x - \alpha = t^2$ **or** $x - \beta = t^2$**.**

Self Practice Problems

(20) Evaluate : $\int \sqrt{\frac{x-3}{x-4}}\,dx$

(21) Evaluate : $\int \frac{dx}{[(x-1)(2-x)]^{3/2}}$

(22) Evaluate : $\int \frac{dx}{[(x+2)^8(x-1)^6]^{1/7}}$

Ans. (20) $\sqrt{(x-3)(x-4)} + \ell n\left(\sqrt{x-3}+\sqrt{x-4}\right) + C$ (21) $2\left(\sqrt{\frac{x-1}{2-x}} - \sqrt{\frac{2-x}{x-1}}\right) + C$

(22) $\frac{7}{3}\left(\frac{x-1}{x+2}\right)^{1/7} + C$

Integration of trigonometric functions

(i) $\int \frac{dx}{a + b\sin^2 x}$ **OR** $\int \frac{dx}{a + b\cos^2 x}$ **OR** $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply Nr & Dr by sec² x & put tan x = t.

(ii) $\int \frac{dx}{a + b\sin x}$ **OR** $\int \frac{dx}{a + b\cos x}$ **OR** $\int \frac{dx}{a + b\sin x + c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put tan $\frac{x}{2}$ = t

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}$ **dx.**

Express Nr ≡ A(Dr) + B (Dr) + C & proceed.

Example # 36 : Evaluate: $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

Solution : Let I = $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$

Putting sinx = $\frac{2\tan x/2}{1+\tan^2 x/2}$ and , cos x = $\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$,

we get

$$I = \int \frac{1+\frac{2\tan x/2}{1+\tan^2 x/2}}{\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)\left(1+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}dx = \int \frac{(1+\tan^2 x/2+2\tan x/2)(1+\tan^2 x/2)}{2\tan x/2(1+\tan^2 x/2+1-\tan^2 x/2)}dx$$

$$= \int \frac{(1+\tan\ x/2)^2\sec^2 x/2}{4\tan x/2}dx = \int \frac{1+t^2+2t)}{2t}dt\text{, where t = tan }\frac{x}{2}$$

$$= \frac{1}{2}\int\left(\frac{1}{t}+t+2\right)dt = \frac{1}{2}\left[\ell n|t|+\frac{t^2}{2}+2t\right]+C = \frac{1}{2}\left[\ell n|\tan x/2|+\frac{\tan^2 x/2}{2}+2\tan x/2\right] + C$$

Example # 37 : Evaluate : $\int \frac{dx}{\sin x + \sqrt{3}\cos x}$

Solution : Let 1 = rcosθ and $\sqrt{3}$ = r sinθ ⇒ r = $\sqrt{(1)^2+(\sqrt{3})^2}$ = 2

tanθ = $\sqrt{3}$ ⇒ θ = π/3

$$\therefore \int \frac{dx}{\sin x+\sqrt{3}\cos x} = \frac{1}{r}\int \frac{dx}{\sin x\cos\theta+\cos x\sin\theta} = \frac{1}{r}\int \frac{dx}{\sin(x+\theta)}$$

$$= \frac{1}{r}\int \text{cosec}(x+\theta)dx = \frac{1}{r}\ \ell n\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C = \frac{1}{2}\ \ell n\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$$

Example # 38 : Evaluate : $\int \frac{3\cos x+2}{\sin x+2\cos x+3}$ dx

Solution : We have,

$$I = \int \frac{3\cos x+2}{\sin x+2\cos x+3}\ dx$$

Let 3 cos x + 2 = λ (sin x + 2 cos x + 3) + μ (cos x – 2 sin x) + ν

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2 \qquad \Rightarrow \qquad \lambda = \frac{6}{5},\ \mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

$$\therefore \qquad I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}\,dx$$

$$\Rightarrow \qquad I = \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx + \nu \int \frac{1}{\sin x + 2\cos x + 3}\,dx$$

$$\Rightarrow \qquad I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu I_1$$

where $I_1 = \int \frac{1}{\sin x + 2\cos x + 3}\,dx$

Putting, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3}\,dx = \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)}\,dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}\,dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2} = dt$ or $\sec^2\frac{x}{2}\,dx = 2\,dt$, we get

$$I_1 = \int \frac{2dt}{t^2 + 2t + 5} = 2\int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$$

Hence, $I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Example # 39 : Evaluate : $\int \frac{dx}{1+3\cos^2 x}$

Solution : Multiply Nr. & Dr. of given integral by $\sec^2 x$, we get

$$I = \int \frac{\sec^2 x\ dx}{\tan^2 x + 4} = \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$$

Self Practice Problems :

(23) Evaluate : $\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\,dx$

Ans. (23) $\frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + C$

Integration of type $\int \sin^m x.\ \cos^n x\ dx$

Case - I

If m and n are even natural number then converts higher power into higher angles.

Case - II

If at least one of m or n is odd natural number then if m is odd put cosx = t and vice-versa.

Case - III

When m + n is a negative even integer then put tan x = t.

Example # 40 : Evaluate : $\int \cos^5 x \sin^4 x dx$

Solution : Let I = $\int \cos^5 x \sin^4 x dx$ put sinx = t $\Rightarrow$ cosx dx = dt

$\Rightarrow$ $I = \int (1-t^2)^2 . t^4 . dt = \int (t^4 - 2t^2 + 1)\ t^4\, dt = \int (t^8 - 2t^6 + t^4)\ dt$

$= \frac{t^9}{9} - \frac{2t^7}{7} + \frac{t^5}{5}$ + C , where t = sinx

Example # 41 : Evaluate : $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx$

Solution : We have ,

$I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx = \int \frac{1}{\cos^{4/3} x \sin^{8/3} x} dx = \int \cos^{-4/3} x \quad \sin^{-8/3} xdx$

divide N^r and D^r by $\cos^4 x$

$= \int \frac{\sec^4 x}{\tan^{8/3} x} dx = \int \frac{(1+\tan^2 x)}{\tan^{8/3} x} \sec^2 x dx = \int \frac{1+\tan^2 x}{\tan^{8/3} x} d(\tan x) = \int \frac{1+t^2}{t^{8/3}} dt$ where t = tan x

$= \int (t^{-8/3} + t^{-2/3}) dt = \frac{-3}{5} t^{-5/3} + 3t^{1/3} + C = \frac{-3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C$

Example # 42 : Evaluate : $\int \sin^4 x \cos^2 x dx$

Solution : $\int \sin^4 x \cos^2 x dx = \frac{1}{8} \int 4 \sin^2 x \cos^2 x . 2 \sin^2 x dx = \frac{1}{8} \int \sin^2 2x\ (1 - \cos 2x) dx$

$= \frac{1}{8} \int \sin^2 2x dx - \frac{1}{8} \int \sin^2 2x \ \cos 2x \ dx = \frac{1}{16} \int (1 - \cos 4x)\ dx - \frac{1}{48} (\sin 2x)^3$

$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{1}{48} (\sin 2x)^3 + C$

Reduction formula of $\int \tan^n x \ dx$, $\int \cot^n x \ dx$, $\int \sec^n x \ dx$, $\int \operatorname{cosec}^n x \ dx$

1. $I_n = \int \tan^n x \ dx = \int \tan^2 x \ \tan^{n-2} x \ dx = \int (\sec^2 x - 1)\ \tan^{n-2} x\, dx$

$\Rightarrow$ $I_n = \int \sec^2 x \tan^{n-2} x\ dx - I_{n-2}$ $\Rightarrow$ $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, $n \ge 2$

2. $I_n = \int \cot^n x \ dx = \int \cot^2 x \ . \ \cot^{n-2} x \ dx = \int (\operatorname{cosec}^2 x - 1) \cot^{n-2} x \ dx$

$\Rightarrow I_n = \int \operatorname{cosec}^2 x \cot^{n-2} x \ dx - I_{n-2}$ $\Rightarrow$ $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$, $n \ge 2$

3. $I_n = \int \sec^n x \ dx = \int \sec^2 x \ \sec^{n-2} x \ dx$

$\Rightarrow$ $I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2)\ \sec^{n-3} x . \sec x \tan x\, dx.$

$\Rightarrow$ $I_n = \tan x \sec^{n-2} x - (n-2)(\sec^2 x - 1) \sec^{n-2} x\, dx$

$\Rightarrow$ $(n-1)\, I_n = \tan x \sec^{n-2} x + (n-2)\, I_{n-2}$ $\Rightarrow I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

4. $I_n = \int \operatorname{cosec}^n x \ dx = \int \operatorname{cosec}^2 x \ \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad I_n = -\cot x \operatorname{cosec}^{n-2} x + \int (\cot x)(n-2)\ (-\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot x)\ dx$

$\Rightarrow \quad -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \cot^2 x \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int (\operatorname{cosec}^2 x - 1)\ \operatorname{cosec}^{n-2} x \ dx$

$\Rightarrow \quad (n-1)\ I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2)\ I_{n-2}$

$\Rightarrow \quad I_n = \dfrac{\cot x \operatorname{cosec}^{n-2} x}{-(n-1)} + \dfrac{n-2}{n-1}\ I_{n-2}$

Example # 43 : Obtain the reduction formula for $\int \cos^n x dx$

Solution : Let $I_n = \int \cos^n x dx$

$I_n = \int \cos x \ (\cos x)^{n-1} dx$

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$I_n = (\sin x)(\cos x)^{n-1} - \int (n-1)(\cos x)^{n-2}(-\sin x)\sin x dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1) \int (\cos x)^{n-2}(1-\cos^2 x)dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1) \int (\cos x)^{n-2} dx - (n-1)\int (\cos x)^n dx$

$I_n = (\sin x)(\cos x)^{n-1} + (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1)I_n = (\sin x)(\cos x)^{n-1} + (n-1)I_{n-2}$

$I_n = \frac{(\sin x)(\cos x)^{n-1}}{n} + \frac{(n-1)}{n} I_{n-2}$, $n \geq 2$

Self Practice Problems :

(24) Deduce the reduction formula for $I_n = \int \frac{dx}{(1+x^4)^n}$ and Hence evaluate $I_2 = \int \frac{dx}{(1+x^4)^2}$.

(25) If $I_{m,n} = \int (\sin x)^m (\cos x)^n dx$ then prove that

$I_{m,n} = \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n} . I_{m,n-2}$

Ans. (24) $I_n = \frac{x}{4(n-1)(1+x^4)^{n-1}} + \frac{4n-5}{4(n-1)} I_{n-1}$

$I_2 = \frac{x}{4 \ (1+x^4)} + \left(\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \ \ell n \left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)\right) + C$