INTEGRALS

INTEGRATION BY PARTIAL FRACTIONS

Integration of rational algebraic functions by using partial fractions :

(i) Partial Fractions :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x.

Let degree of f(x) < degree of g(x) [if it is not so, divide f(x) by g(x) until the degree of numerator becomes less than that of denominator] Apply the concept of partial fractions as below:

CASE I :

When denominator is expressible as the product of non-repeating linear factors.

Let $g(x) = (x - a_1)(x - a_2) \ldots\ldots(x - a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} \equiv \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots\ldots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, \ldots\ldots A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, \ldots\ldots, a_n$.

CASE II :

When the denominator g(x) is expressible as the product of the linear factors such that some of them are repeating.

Example:

$\frac{1}{g(x)} = \frac{1}{(x-a)^k (x-a_1)(x-a_2)\ldots\ldots(x-a_r)}$ this can be expressed as

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \ldots\ldots + \frac{B_r}{(x-a_r)}$$

Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by equating the coefficient of same power of x. The following example illustrate the procedure.

CASE III :

When some of the factors of denominator g(x) are quadratic but non-repeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the ty $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$ The following example illustrates the procedure.

CASE IV :

When some of the factors of the denominator g(x) are quadratic and repeating fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c}+\frac{A_1}{ax^2+bx+c}\right\} + \left\{\frac{A_1(2ax+b)}{\left(ax^2+bx+c\right)^2}+\frac{A_2}{\left(ax^2+bx+c\right)^2}\right\} + \ldots\ldots + \left\{\frac{A_{2k-1}(2ax+b)}{\left(ax^2+bx+c\right)^k}+\frac{A_{2k}}{\left(ax^2+bx+c\right)^k}\right\}$$

Ex.1 Evaluate $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Sol. Let $\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$

$\Rightarrow \quad \frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$

Putting x = 1, –6A = 1 $\Rightarrow$ $A = -\frac{1}{6}$

Putting x = 3, 10C = 5 $\Rightarrow$ $C = \frac{1}{2}$

Putting x = –2, 15B = 5 $\Rightarrow$ $B = -\frac{1}{3}$

So $= -\frac{1}{6}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x-3}\,dx$

$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log_e|x+2| + \frac{1}{2}\log_e|x-3| + C$

Ex.2 Resolve $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Sol. Here the given function is an improper rational function (i.e. degree of numerator > degree of denominator) .On dividing we get

$\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 + \frac{(-x+4)}{(x^2-5x+6)}$ ....(i)

we have, $\frac{-x+4}{x^2-5x+6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$, then

$-x + 4 = A(x-3) + B(x-2)$ ....(ii)

Putting $x - 3 = 0$ or $x = 3$ in (ii), we get

$1 = B(1) \quad \Rightarrow \quad B = 1.$

Putting $x - 2 = 0$ or $x = 2$ in (ii), we get

$2 = A(2-3) \Rightarrow A = -2$

$\therefore \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2} + \frac{1}{x-3}$

Hence $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 - \frac{2}{x-2} + \frac{1}{x-3}$

Ex.3 Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate $\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

Sol. Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} + \frac{A_4}{x+2}$

$\Rightarrow \quad 3x - 2 = A_1(x-1)(x+1)(x+2) + A_2(x+1)(x+2)$

$+ A_3(x-1)^2(x+2) + A_4(x-1)^2(x+1)$ ....(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i) we get

$1 = A_2(1+1)(1+2) \Rightarrow A_2 = \frac{1}{6}$

Putting $x + 1 = 0$ or, $x = -1$ in (i) we get

$-5 = A_3(-2)^2(-1+2) \Rightarrow A_3 = -\frac{5}{4}$

Putting $x + 2 = 0$ or, $x = -2$ in (i) we get

$$-8 = A_4(-3)^2(-1) \Rightarrow A_4 = \frac{8}{9}$$

Now equating coefficient of x^3 on both sides, we get $0 = A_1 + A_3 + A_4$

$$\Rightarrow \quad A_1 = -A_3 - A_4 = \frac{5}{4} - \frac{8}{9} = \frac{13}{36}$$

$$\therefore \frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{13}{36(x-1)} + \frac{1}{6(x-1)^2} - \frac{5}{4(x+1)} + \frac{8}{9(x+2)}$$

and hence $\displaystyle\int \frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

$$= \frac{13}{36}\lambda n|x-1| - \frac{1}{6(x-1)} - \frac{5}{4}\lambda n|x+1| + \frac{8}{9}\lambda n|x+2| + C$$

Ex.3 Evaluate $\displaystyle\int \frac{x^2}{(x^2+4)(x^2+1)}dx$

Sol. $\displaystyle\int \frac{x^2}{(x^2+4)(x^2+1)}dx$

$$= \frac{1}{3}\int\left[\frac{4}{x^2+4} - \frac{1}{x^2+1}\right]dx$$

$$= \frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}x + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}\tan^{-1}x + C$$

Ex.4 Resolve $\dfrac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\dfrac{2x-3}{(x-1)(x^2+1)^2} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$. Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting $x = 1$ in (i), we get $-1 = A(1 + 1)^2$ A = –

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0$, $C - B = 0$,

$2A + B - C + D = 0$,

$C + E - B - D = 2$ and $A - C - E = -3$.

Putting A = – $\frac{1}{4}$ and solving these equations, we get

B = $\frac{1}{4}$ = C, D = $\frac{1}{4}$ and E = $\frac{5}{2}$

$\therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$

Ex.5 Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$.

Then, $2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ...(i)

Putting x – 1 = 0 or, x = 1 in (i), we get $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting x = 0 in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting x = – 1 in (i), we get $-2 = A + 2B - 2C$.

$\Rightarrow \quad -2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3} \cdot \frac{1}{x-1} + \frac{(-2/3)x + 2/3}{x^2+x+1}$ or $\frac{2x}{x^3-1}$

$= \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$