INTEGRALS

INTEGRALS OF SOME PARTICULAR FUNCTIONS

Integration of type

$\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

Ex.1 Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Sol. We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\ dx = \int \sqrt{(x+1)^2+2^2}$

$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}.(2)^2 \lambda n\,|(x+1) + \sqrt{(x+1)^2+2^2}\,| + C$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\lambda n\,|(x+1) + \sqrt{x^2+2x+5}\,| + C$

Ex.2 Evaluate : $\int \frac{1}{x^2-2x+3}\ dx$

Sol. $I = \int \frac{1}{x^2-2x+3}\ dx$

$= \int \frac{1}{(x-1)^2+2}\ dx$

$= \int \frac{1}{(x-1)^2+(\sqrt{2})^2}\ dx$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$

Ex.3 Evaluate : $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx$

Sol. $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx$

$$= \int \frac{1}{\sqrt{-\{x^2 - 8x - 33\}}}\, dx$$

$$= \int \frac{1}{\sqrt{-\{x^2 - 8x + 16 - 49\}}}\, dx$$

$$= \int \frac{1}{\sqrt{-\{(x-4)^2 - 7^2\}}}\, dx$$

$$= \int \frac{1}{\sqrt{7^2 - (x-4)^2}}\, dx$$

$$= \sin^{-1}\left(\frac{x-4}{7}\right) + C$$

Integration of type

$$\int \frac{px+q}{ax^2+bx+c}\, dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx,\ \int (px+q)\sqrt{ax^2+bx+c}\, dx$$

Express $px + q = A$ (differential co–efficient of denominator) $+ B$.

Ex.4 Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\, dx$

Sol. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\, dx$

$$= \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\, dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+1}}\, dx - \int \frac{1}{\sqrt{x^2+4x+1}}\, dx$$

$$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2}}\, dx,$$

where $t = (x^2 + 4x + 1)$ for I^{st} integral

$$= 2\sqrt{t} - \lambda n \mid (x+2) + \mid + \sqrt{x^2+4x+1}\ C$$

$$= 2\sqrt{x^2+4x+1} - \lambda n \ |x+2+\sqrt{x^2+4x+1}\ | + C$$

Ex.5 Evaluate : $\int (x-5)\sqrt{x^2+x}\, dx$

Sol. Let $(x - 5) = \lambda.\frac{d}{dx}(x^2 + x) + \mu$.

Then, $x - 5 = \lambda (2x + 1) + \mu$.

Comparing coefficients of like powers of x, we get

$1 = 2\lambda$ and $\lambda + \mu = -5 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\frac{11}{2}$

Hence, $\int (x-5)\sqrt{x^2+x}\, dx = \int \left(\frac{1}{2}(2x+1) - \frac{11}{2}\right)\sqrt{x^2+x}\, dx$

$$= \int \frac{1}{2}(2x+1)\sqrt{x^2+x}\, dx - \frac{11}{2}\int \sqrt{x^2+x}\, dx$$

$$= \frac{1}{2}\int \sqrt{t}\, dt - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dx \quad \text{(where } t = x^2 + x \text{ for first integral)}$$

$$= \frac{1}{2}.\frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right.\right.$$

$$\left.\left. - \frac{1}{2}.\left(\frac{1}{2}\right)^2 \lambda n\left[\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right\}\right] + C$$

$$= \frac{1}{3}t^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right|\right] + C$$

$$= \frac{1}{3}(x^2 + x)^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x}\right|\right] + C$$