# INTEGRALS

## DEFINITE INTEGRAL

Here we shall define integration as a process of summation or definite integral as the limit of a sum. Then we discuss some properties of definite integral. The concept of definite integral is then used to find the area enclosed by certain curves.

## DEFINITE INTEGRAL

Let f(x) be a continuous real valued function defined on [a, b] such that . Then

$\int f(x)dx = F(x) + C$. Then $\int_a^b f(x)dx = F(b) - F(a)$ called definite integral of f(x) in [a, b].

## Remark :

1. If f(x) is discontinuous at x = a and continuous at x = b

   then $\int_a^b f(x)dx = F(b) - \lim_{x \to a+} F(x)$

2. If f(x) is discontinuous at x = b and continuous at x = a then

   $\int_a^b f(x)dx = \lim_{x \to b-} F(x) - F(a)$

3. If f (x) is discontinuous at x = a and x = b then

   $\int_a^b f(x)dx = \lim_{e \to 0-} \int_{a+e}^{b-e} f(x)dx$ or $\lim_{x \to b-} F(x) - \lim_{x \to a+} F(x)$

4. If f (x) is discontinuous at x = c (a < c < b) then

   $\int_a^b f(x)dx = \lim_{e \to 0} \int_a^{c-e} f(x)dx + \lim_{e \to 0} \int_{c+e}^{b} f(x)dx$

Note that even if f (x) is not defined at x = a or x = b or at both, $\int_a^b f(x)dx$ can be evaluated.

a and b are called lower and upper limits of integration respectively. If we make change in variable (i.e. substitution) then limit of integration should be changed accordingly.

**Ex.1** Evaluate $\int_0^4 \left(x - 2\sqrt{x} + x^2\right) dx$

**Sol.** $\int_0^4 \left(x - 2\sqrt{x} + x^2\right) dx - \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} + \frac{x^3}{3}\right]_0^4$

$-\left(8 - \frac{32}{3} + \frac{64}{3}\right) - (0)$

$= \frac{56}{3}$

## Definite Integral as the Limit of a Sum

(i) Express the given series in the form of $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) The limit when n $\to \infty$ is its sum $\lim_{n \to 0} \sum_{r=0}^{n-1} \frac{1}{n} \cdot f\left(\frac{r}{n}\right)$

Replace r/n by x, 1/n by dx and $\lim_{n \to \infty} \sum$ by the sign of integration $\int$

(iii) The lower and upper limits of integration will be the value of r/n for the first and last term (or the limits of these values respectively).

## Some Important Formulae

1. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

2. $\sum_{n=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{n=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

4. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) - \frac{\sin n\beta/2}{\sin\beta/2} \sin\left[\frac{1^{\text{st}}\ \text{angle+ lastangle}}{2}\right]$

5. $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+(n-1)\beta)$

$$=\frac{\sin n\beta/2}{\sin\beta/2}\cos\left[\frac{1^{\text{st}}\text{ angle}+\text{ last angle}}{2}\right].$$

6. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots\ldots\ \text{to}\,00=\log_6 2$

7. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}\cdots\cdots\cdots=\frac{\pi^2}{12}.$

8. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots=\frac{\pi^2}{6}$

9. $1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\cdots=\frac{\pi^2}{8}$

10. $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\cdots\cdots\cdots=\frac{\pi^2}{24}.$

11. $\cos\theta=\frac{e^{18}+e^{-18}}{2}\cdot\sin\theta=\frac{e^{B}-e^{-18}}{2}$

12. $\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}.$

**Ex.2** Compute the following limits (using definite integral)

(1) $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$

(2) $\lim_{n\to\infty}\frac{\pi}{2n}\left(1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots+\cos\frac{(n-1)\pi}{2n}\right)$

**Sol.** (1) $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right)$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{1+\frac{r}{n}}$$

$$=\int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{\frac{3}{2}}\right]_0^1 = \frac{2^{3/2}}{\frac{3}{2}} - \frac{1}{3} - \frac{2}{3}\{2\sqrt{2}-1\}$$

(2) $\lim_{n\to\infty} \frac{\pi}{2n}\left[1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots+\cos\frac{(n-1)\pi}{2n}\right]$

$$=\lim_{n\to\infty} \frac{(n/2)}{n}\sum_{r=0}^{n-1}\cos\left(r\left(\frac{\pi/2}{n}\right)\right)$$

[Note here interval$[0,\pi/2]$ is divided in $n$ equal parts]

$$=\int_0^{\pi/2}\cos x\,dx - [\sin x]_0^{\pi/2} - 1$$