DIFFERENTIAL EQUATIONS

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A linear differential equation has the following characteristics:

1. The dependent variable and its derivative are of the first degree and not multiplied together.
2. All derivatives should be in polynomial form.
3. The order of the derivatives may be more than one.

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^m y}{dx^m}+P_1(x)\frac{d^{m-1}y}{dx^{min-1}}+\ldots\ldots\ldots\ldots\ldots\ldots+P_{m-1}(x)\frac{dy}{dx}+P_m(x)y=\phi(x)$$

Where $P_0(x)$, $P_1(x)$ ....$P_m(x)$ are called the coefficients of the differential equation.

The coefficients of the differential equation are denoted as $P_0(x)$, $P_1(x)$ ....$P_m(x)$.

Note: While a linear differential equation is always of the first degree, it's important to note that not every differential equation of the first degree is linear.

E.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=$ is not linear, though its degree is 1.

$\frac{dy}{dx}+y^2\sin x=\ln x$ is not a Linear differential equation.

Ex.1 Which of the following equations is linear and which one is nonlinear?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$

(D) $x\frac{dy}{dx}+y^2=\sin x$ (E) $\frac{dy}{dx}=\cos x$ (F) $\frac{d^2y}{dx^2}+y=0$

(G) $dx+dy=0$ (H) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Sol. None Linear (A), (C), (D) , (H)
Linear (B), (E), (F),(G)

Linear differential equations of first order

The differential equation $\frac{dy}{dx}+Py=Q$ is linear in y.

(Where P and Q are functions solely dependent on x.)

Integrating Factor (I.F.) :

It is an expression that, when multiplied by a differential equation, transforms it into an exact form.

The integrating factor for a linear differential equation is = $g(x)=e^{\int Pdx}$ After multiplying the above equation by (ignoring the constant of integration),
I.F it becomes;

$$\frac{dy}{dx}\cdot e^{\int Pdx}+Py\cdot e^{\int Pdx}=Q\cdot e^{\int Pdx}$$

$$\frac{d}{dx}\left(y\cdot e^{\int Pdx}\right)=Q\cdot e^{\int Pdx}$$

$$e^{\int Pdx}=\int Q\cdot e^{\int Pdx}+C$$

At times, the differential equation becomes linear when x is considered the dependent variable and y as the independent variable. In such cases, the differential equation takes the following form:

$$\frac{dx}{dy}+P_1x=Q_1$$

Here, P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1 dy}$

Ex.2 Find $\left(1+y^2\right)+\left(x-e^{tn^{-1}y}\right)\frac{dy}{dx}=0$

Sol. The differential equation can be expressed as:

$$\left(1+y^2\right)\frac{dx}{dy}+x=e^{\tan^{-1}y}$$

$$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

So, solution is $$xe^{\tan^{-1}y}=\int\frac{e^{\tan-1}ye^{\tan^{-1}y}}{1+y^2}dy$$

Let, $$e^{\tan-1}y=t$$

$$\frac{e^{\tan-1}y}{1+y^2}dy=dt$$

$$xe^{\tan^{-1}y}=\int tdt$$

Putting $$e^{\tan-1}y=t$$

Or $$xe^{\tan-1}y=\frac{t^2}{2}+\frac{c}{2}$$

$$2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c$$

Ex.3 Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2x}{1+x^3}$

Sol. $$\frac{dy}{dx}+Py=Q$$

$$P=\frac{3x^2}{1+x^3}$$

$$F=e^{\int P\cdot dx}=e^{\int\frac{3x^2}{1+x^3}dx}=e^{\ln\left(1+x^3\right)}=1+x^3$$

General solution is

$$y(\ IF)=\int Q(\ IF)\cdot dx+c$$

$$y\left(1+x^3\right)=\int\frac{\sin^2x}{1+x^3}\left(1+x^3\right)dx+c$$

$$y\left(1+x^3\right)=\int\frac{1-\cos2x}{2}dx+c$$

$$y\left(1+x^3\right)=\frac{1}{2}x-\frac{\sin2x}{4}+c$$

Ex.4 Evaluate $x\ln x\frac{dy}{dx}+y=2\ln x$

Sol. $$\frac{dy}{dx}+y=\frac{2}{x}$$

$$P = \frac{1}{x / \ln x}, Q = \frac{2}{x}$$

$$IF = e^{\int P.dx} = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x$$

General solution is

$$y \cdot (\ln x) = \int \frac{2}{y} \ln x \cdot dx + c$$

$$y(\ln x) = (\ln x)^2 + c$$

Equations reducible to linear form

(A) By change of variable.

Frequently, a differential equation can be transformed into a linear form through a suitable substitution of the non-linear term.

Ex.5 Solve: $\frac{dy}{dx} = \cos x\left(\sin x - y^2\right)$

Sol. The provided differential equation can be transformed into a linear form through a change of variable by an appropriate substitution.

Substituting $y^2 = z$

$$2y\frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x \cdot z = \sin x \cos x$$

$\frac{dz}{dx} + 2\cot x \cdot z = 2\cos x$ which is linear in $\frac{dz}{dx}$

$$IF = e^{\int 2\cos x dx} = e^{2\min x} = \sin^2 x$$

General solution is

$$z. \sin^2 x = \int 2\cos x \cdot \sin^2 x \cdot dx + c$$

$$y^2 \sin^2 x = \frac{2}{3}\sin^3 x + c$$

(B) Bernoulli's equation

Equations of the form $\frac{dy}{dx} + Py = Q \cdot y^n, n \neq 0$ and $n \neq 1$ where P and Q are functions of x, is called Bernoulli's equation.

Equations in the form $\frac{dy}{dx}+Py=Q\cdot y^{n}, n\neq 0$ and $n\neq 1$ where P and Q are functions of x, are referred to as Bernoulli's equations.

e.g. $$2\sin x\frac{dy}{dy}-y\cos x=xy^{3}e^{x}$$

On dividing by y^n

We get $$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q$$

Let $$y^{n-1}=t,$$

So that $$(-n+1)y^{-1}\frac{dy}{dx}=\frac{dt}{dx}$$

Then equation becomes $\frac{dt}{dx}+P(1-n)t=Q(1-n)$ linear with t as a dependent variable.

Ex.6 Find $\frac{dy}{dx}-\frac{y}{x}=\frac{y^{2}}{x^{2}}$ (Bernoulli's equation)

Sol. Dividing both sides by y^2

$$\frac{1}{y^{2}}\frac{dy}{dx}-\frac{1}{xy}=\frac{1}{x^{2}}$$

Putting $$\frac{1}{y}=t$$

$$-\frac{1}{y^{2}}\frac{dy}{dx}=\frac{dt}{dx}$$

Differential equation (i) becomes,

$$-\frac{dt}{dx}-\frac{t}{x}=\frac{1}{x^{2}}$$

$\frac{dt}{dx}+\frac{t}{x}=-\frac{1}{x^{2}}$ which is linear differential equation in $\frac{dt}{dx}$

$$=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$t\cdot x=\int-\frac{1}{x^{2}}\cdot xdx+c$$

$$tx=-\ln x+c$$

$$\frac{x}{y}=-\ln x+c$$

Ex.7 Determine the solution for the given differential equation: $\frac{dy}{dx} - y\tan x = -y^2 \sec x$

Sol.
$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

$$\frac{1}{y} = v; \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{-dv}{dx} - v\tan x = -\sec x$$

$$\frac{dv}{dx} + v\tan x = \sec x,$$

Here, $P = \tan x, Q = \sec x$

I.F. $= e^{\int \mathrm{sanxdx}} = |\sec x|\ v\ |\sec x| = \int \sec^2 x dx + c$

Hence the solution is $y^{-1}|\sec x| = \tan x + c$