DIFFERENTIAL EQUATIONS

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A function f(x, y) is considered a homogeneous function of degree n if the substitution $x = \lambda x$, $y = \lambda y$, , $\lambda > 0$ results in the equality:

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

The degree of homogeneity, denoted by 'n', can take any real number.

Ex.1 Find the degree of homogeneity for the given function.

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = \left(\dfrac{x^{\frac{3}{2}} + y^{\frac{3}{2}}}{(x+y)}\right)$ (iii) $f(x,y) = \sin\left(\dfrac{x}{y}\right)$

Sol. (i)

$$f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2 \left(x^2 + y^2\right)$$
$$= \lambda^2 f(x,y)$$

Degree of homogeneity $\rightarrow 2$

(ii)

$$f(\lambda x, \lambda y) = \frac{\lambda^{3/2} x^{3/2} + \lambda^{3/2} y^{3/2}}{\lambda x + \lambda y}$$
$$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$$

Degree of homogeneity $\rightarrow$ ½

(iii)

$$f(\lambda x, \lambda y) = \sin\left(\frac{\lambda x}{\lambda y}\right)$$
$$= \lambda^{\circ} \sin\left(\frac{x}{y}\right)$$
$$= \lambda^{\circ} f(x,y)$$

Degree of homogeneity $\rightarrow 0$

Ex.2 Ascertain whether each of the following functions is homogeneous or not.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \dfrac{xy}{x+y^2}$ (iii) $f(x,y) = \sin xy$

Sol. (i) $= \lambda^2 x^2 - \lambda^2 xy$

$= \lambda^2 \left(x^2 - xy\right) = \lambda^2 f(x, y)$ Homogeneous.

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x, y)$ Not homogeneous.

(i) $f(\lambda x, \lambda y) = \sin\left(\lambda^2 xy\right) \neq \lambda^{n_f}(x, y)$ Not homogeneous.

Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$

Where f (x,y) and g(x,y) are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting y = vx,

A function F(x, y) where f(x, y) and g (x, y) are homogeneous functions of x and y and of the same degree, is considered homogeneous. Such equations can be solved by substituting y = vx.

(i) So that the dependent variable y is changed to another variable v.

As f(x, y) and g(x, y) are homogeneous functions of the same degree, denoted as n, they can be expressed as:

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right)$$

$$g(x,y) = x^n g_1\left(\frac{y}{x}\right)$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

The given differential equation, therefore, becomes

$$v + x$$

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)}$$

$$\frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

So that the variables v and x are now separable.

In certain cases, homogeneous equations can be resolved by substituting (x = vy) or by employing polar coordinate substitution.

Ex.3 Find the solution for the differential equation $(x^2 - y^2)\,dx + 2xydy = 0$, with the initial condition $y = 1$ when $x = 1$.

Sol.

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

$$y = ux$$

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{1 - v^2}{2v}$$

$$\int \frac{2v}{1 + v^2}dv = -\int \frac{dx}{x}$$

$$\text{In}\left(1 + v^2\right) = -\text{In}x + c$$

$$x=1,\ y=1,\quad v=1$$

$$\text{In } 2 = c$$

$$\text{In}\left\{\left(1 + \frac{y^2}{x^2}\right).x\right\} = \text{In}2$$

$$x^2 + y^2 = 2x$$

Ex.4 Find the solution to the differential equation: $\left(1 + 2e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Sol. The equation is homogeneous with a degree of 0.

Put $\quad x = vy, \quad dx = vdy + ydv$

Subsequently, the differential equation transforms into

$$\left(1 + 2e^v\right)(vdy + ydv) + 2e^v(1 - v)dy = 0$$

$$\left(v + 2e^v\right)dy + y\left(1 + 2e^v\right)dv = 0$$

$$\frac{dy}{y} + \frac{1 + 2e^v}{v + 2e^v}dv = 0$$

Integrating and replacing v by $\frac{x}{y}$,

We get $\qquad \ln y + \ln\left(v + 2e^V\right) = \ln c$

And $\qquad x + 2ye^{\frac{x}{y}} = c$

(i) Equations reducible to homogeneous form

An equation of the form where $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$ can be reduced to homogeneous form by changing the variables x, y to u, v as $x=u+h, y=v+k$

Where h, k are constants chosen to make the given equation homogeneous.

We have $$\frac{dy}{dx}=\frac{dv}{du}$$

The equation becomes,

$$\frac{dv}{du}=\frac{a_1u+b_1v+\left(a_1h+b_1k+c_1\right)}{a_2u+b_2v+\left(a_2h+b_2k+c_2\right)}$$

Let, h and k be chosen so as to satisfy the equation

$$a_1\ h+b_1k+c_1=0 \qquad \text{... (i)}$$

$$a_2\ h+b_2k+c_2=0 \qquad \text{... (ii)}$$

Solve for h and k from (i) and (ii)

Now, $$\frac{du}{dv}=\frac{a_1u+b_1v}{a_2u+b_2v}$$

Is a homogeneous equation and can be solved by substituting v = ut.

Ex.5 Solve the differential equation $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

Sol. Let $$x = x + h, y = Y + k$$

$$\frac{dy}{dX}=\frac{d}{dX}(Y+k)$$

$$\frac{dy}{dX}=\frac{dY}{dX}$$

$$\frac{dx}{dX}=1+0$$

On dividing (i) by (ii)

$$\frac{dy}{dx}=\frac{dY}{dx}$$

$$\frac{dY}{dX}=\frac{X+h+2\left(Y+k\right)-5}{2X+2h+Y+k-4}$$

$$=\frac{X+2Y+\left(h+2k-5\right)}{2X+Y+\left(2h+k-4\right)}$$

h & k are such that

$$h + 2k - 5 = 0$$
$$2h + k - 4 = 0$$
$$h = 1, k = 2$$

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ Which is homogeneous differential equation.

Now, substituting $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$\int \frac{2+v}{1-v^2}dv = \int \frac{dX}{X}$$

$$\int\left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right)dv = \text{In}X + c$$

$$\frac{1}{2}\ln(v+1) - \frac{3}{2}\ln(1-v) = \ln X + c$$

$$\ln\left|\frac{v+1}{(1-v)^3}\right| = \ln X^2 + 2c$$

$$\frac{(Y+X)}{(X-Y)^3}\frac{X^2}{X^2} = e^{2c}$$

$$X + Y = c'(X-Y)^3$$

$$e^{2c} = c^1$$

$$x - 1 + y - 2 = c'(x - 1 - y + 2)^3$$

$$x + y - 3 = c'(x - y + 1)^3$$

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Ex.6 Solve: $(x + y)dx + (3x + 3y - 4)\,dy = 0$

Sol. Let

$$t = x + y$$
$$dy = dt - dx$$
$$tdx + (3t - 4)\,(dt - dx) = 0$$

$$2dx+\left(\frac{3t-4}{2-t}\right)dt=0$$

$$2dx-3dt+\frac{2}{2-t}dt=0$$

Integrating and replacing t by x + y, we get

$$2x - 3t - 2[\ln|(2-t)|] = c_1$$

$$2x - 3(x+y) - 2[\ln|(2-x-y)|] = c_1$$

$$x + 3y + 2\ln|(2-x-y)| = c$$

Ex.7 Evaluate $\frac{dy}{dx}=\frac{2x+3y-1}{4x+6y-5}$

Sol. Putting $u = 2x + 3y$

$$\frac{1}{3}\left(\frac{du}{dx}-2\right)=\frac{u-1}{2u-5}$$

$$\frac{du}{dx}=\frac{3u-3+4u-10}{2u-5}$$

$$\int\frac{2u-5}{7u-13}dx=\int dx$$

$$\frac{2}{7}\int 1\cdot du-\frac{9}{7}\int\frac{1}{7u-13}\cdot du=x+c$$

$$\frac{2}{7}u-\frac{9}{7}\cdot\frac{1}{7}\ln(7u-13)=x+c$$

$$4x+6y-\frac{9}{7}\ln(14x+21y-13)=7x+7c$$

$$-3x+6y-\frac{9}{7}\ln(14x+21y-13)=c'$$

If $a_2 + b_1 = 0$, then a simple cross multiplication and substituting d (xy) for xdy + ydx and integrating term by term, yield the results easily.

Ex.8 Evaluate $\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$

Sol.

$$\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$$

$$2xdy+2ydy+3dy=xdx-2ydx+dx$$

$$(2y+3)dy=(x+1)dx-2(xdy+ydx)$$

On integrating,

$$\int(2y+3)dy=\int(x+1)dx-\int 2\ d(xy)$$

$$y^2+3y=\frac{x^2}{2}+x-2xy+c$$

Ex.9 Find $\frac{dy}{dx}=\frac{x-2y+5}{2x+y-1}$

Sol. Cross multiplying,

$$2xdy+ydy-dy=xdx-2ydx+5dx$$

$$2(xdy+ydx)+ydy-dy=xdx+5dx$$

$$2d(xy)+ydy-dy=xdx+5dx$$

On integrating,

$$2xy+\frac{y^2}{2}-y=\frac{x^2}{2}+5x+c$$

$$x^2-4xy-y^2+10x+2y=c'$$

$$c'=-2c$$