DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE

A first-order and first-degree differential equation belongs to the type:

$$\frac{dy}{dx} + f\left(x, y\right) = 0$$

This expression can also be represented as: Mdx + Ndy = 0, where M and N are functions of x and y.

Solution methods of First Order and First Degree Differential Equations

(A) Separation of Variables

Certain differential equations can be solved using the separation of variables method. This approach is applicable when the differential equation can be expressed in the form A(x) dx + B(y) dy = 0.

Where A (x) is solely a function of 'x' and B(y) is solely a function of 'y'.

A general solution for this is provided by,

$$\int A(x)\,dx + \int B(y)dy = c$$

Where 'c' is the arbitrary constant.

Ex.1 Find the solution to the differential equation (1 + x) y, dx = (y – 1) x , dy.

Sol. The equation can be written as –

$$\left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy$$

$$\int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\text{In } x + x = y - \text{In } y + c$$

$$\text{In } y + \text{In } x = y - x + c$$

$$xy = e^{y-x}$$

Ex.2 Solve the differential equation $xy\frac{dy}{dx}=\frac{1+y^2}{1+x^2}\left(1+x+x^2\right)$.

Sol. The differential equation can be expressed as:

$$xy\frac{dy}{dx}=\left(1+y^2\right)\left(1+\frac{x}{1+x^2}\right)$$

$$\frac{y}{1+y^2}dy=\left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$$

Integrating,

$$\frac{1}{2}\ln\left(1+y^2\right)=\ln x+\tan^{-1}x+c$$

$$\sqrt{1+y^2}=cxe^{\tan^{-1}x}$$

Ex.3 Solve $\frac{dy}{dx}=\left(e^x+1\right)\left(1+y^2\right)$

Sol. The equation can be formulated as:

$$\frac{dy}{1+y^2}=\left(e^x+1\right)dx$$

Integrating both sides, $\tan^{-1}y=e^x+x+c$.

Ex.4 Solve the differential equation $\left(x^3-y^2x^3\right)\frac{dy}{dx}+y^3+x^2y^3=0$

Sol.

$$\left(x^3-y^2x^3\right)\frac{dy}{dx}+y^3+x^2y^3=0$$

$$\frac{1-y^2}{y^3}dy+\frac{1+x^2}{x^3}dx=0$$

$$\int\left(\frac{1}{y^3}-\frac{1}{y}\right)dy+\int\left(\frac{1}{x^3}+\frac{1}{x}\right)dx=0$$

$$\log\left(\frac{x}{y}\right)=\frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right)+c$$

POLAR COORDINATES TRANSFORMATIONS

At times, converting to polar coordinates makes the separation of variables more convenient. In this context, it is helpful to recall the following differentials:

(A) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$

(ii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

(iii) $xdy - ydx = r^2 d\theta$

(B) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $xdx - ydy = rdr$

(ii) $xdy - ydx = r^2\sec\theta\, d\theta$.

Ex.5 Solve the differential equation $xdx + ydy = x(xdy - ydx)$

Sol. Taking $x = r\cos\theta$, $y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$2xdx + 2ydy = 2rdr$$

$$xdx + ydy = rdr \qquad \text{......... (i)}$$

$$\frac{y}{x} = \tan\theta$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$$xdy - ydx = x^2\sec^2\theta \cdot d\theta$$

$$xdy - ydx = r^2 d\theta \qquad \text{........ (ii)}$$

By employing (i) and (ii) in the provided differential equation, it transforms into:

$$rdr = r\cos\theta \cdot r^2 d\theta$$

$$\frac{dr}{r^2} = \cos\theta d\theta$$

$$\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}}=c$$

$$(y+1)^2=c\left(x^2+y^2\right)$$

Ex.6 Solve : $\dfrac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$

Sol.
$$\frac{x+y\dfrac{dy}{dx}}{y-x\dfrac{dy}{dx}}=x^2+2y^2+\frac{y^4}{x^2}$$

$$=\frac{x^4+2x^2y^2+y^4}{x^2}$$

$$\frac{xdx+ydy}{ydx-xdy}=\frac{\left(x^2+y^2\right)^2}{x^2}$$

$$\frac{xdx+ydx}{\left(x^2+y^2\right)^2}=\frac{-\left(xdy-ydx\right)}{x^2}$$

$$\frac{1}{2}\frac{2xdx+2ydy}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

$$\frac{1}{2}\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=-d\left(\frac{y}{x}\right)$$

Now integrating both sides

$$-\frac{1}{2}\frac{1}{x^2+y^2}=-\frac{y}{x}+c$$

$$\frac{y}{x}-\frac{1}{2\left(x^2+y^2\right)}=c$$

Equations Reducible to the Variables Separable form

If a differential equation can be transformed into a separable variables form through an appropriate substitution, it is termed "Reducible to the variables separable type."

Its general form is. $\dfrac{dy}{dx}=f(ax+by+c)a,b\neq 0$. To solve this, put ax + by + c = t.

Ex.7 Solve $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$.

Sol.

$$\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$$

$$x + y = t$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\frac{dt}{dx} - 1 = \cos t - \sin t$$

$$\int \frac{dt}{1 + \cos t - \sin t} = \int dx$$

$$\int \frac{\sec^2 \frac{t}{2} dt}{2\left(1 - \tan \frac{t}{2}\right)} = \int dx$$

$$-\ln\left|1 - \tan\frac{x+y}{2}\right| = x + c$$

Ex.8 Solve $\frac{dy}{dy} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes $\frac{dt}{dx} - 4 = t^2$

$$\frac{dt}{t^2 + 4} = dx$$

(Variables are separated)
Integrating both sides,

$$\int \frac{dt}{4 + t^2} = \int dx$$

$$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x + y + 1}{2}\right) = x + c$$

Ex.9 Solve: $y' = (x + y + 1)^2$

Sol.
$$y' = (x + y + 1)^2 \quad \text{....(i)}$$

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i)

We get,
$$\frac{dt}{dx} = t^2 + 1$$

$$\int \frac{dt}{1+t^2} = \int dx$$

$$\tan^{-1} t = x + C$$

$$t = \tan(x + C)$$

$$x + y + 1 = \tan(x + C)$$

$$y = \tan(x + C) - x - 1$$

EQUATION OF THE FORM

$$\Rightarrow \quad yf\,(xy)dx + xg\,(xy)dy = 0 \quad \text{.......... (i)}$$

The substitution $xy = z$ transforms the differential equation of this structure into a form where the variables are separable.

Let
$$xy = z \quad \text{.......... (ii)}$$

$$dy = \left[\frac{xdz - zdx}{x^2}\right]$$

using equation (ii) \& (iii), equation (i) becomes

$$\frac{z}{x}f(z)dx + xg(z)\left[\frac{xdz - zdx}{x^2}\right] = 0$$

$$\frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0$$

$$\frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0a$$

$$\frac{1}{x}dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$$

Ex.10 Solve $y(xy+1)dx + x\left(1+xy+x^2y^2\right)dy = 0a$

Sol. Let
$$xy = v$$
$$y = \frac{v}{x}$$
$$dy = \frac{xdv - vdx}{x^2}$$

Now, differential equation becomes
$$\frac{v}{x}(v+1)dx + x\left(1+v+v^2\right)\left(\frac{xdv - vdx}{x^2}\right) = 0$$

On solving, we get
$$v^3dx - x\,(1+v+v^2)\,dv = 0$$

Separating the variables & integrating

We get,
$$\int\frac{dx}{x} - \int\left(\frac{1}{v^3}+\frac{1}{v^2}+\frac{1}{v}\right)dv = 0$$
$$\ln x + \frac{1}{2v^2} + \frac{1}{v} - \ln v = c$$
$$2v^2 \ln\left(\frac{v}{x}\right) - 2v - 1 = -2cv^2$$
$$2x^2y^2 \ln n - 2xy - 1 = Kxx^2y^2$$
$$\text{where } K = -2c$$