APPLICATION OF INTEGRALS

INTRODUCTION, AREA UNDER SIMPLE CURVES

Area under the curve :

(i) Curve-tracing :

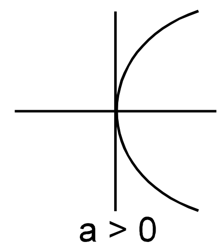
To approximate the shape of a curve, the following expressions are recommended:

(a) Symmetry:

- **Symmetry about x-axis :**

  If all the exponents of 'y' in the equation are even, then the curve (graph) exhibits symmetry about the x-axis.

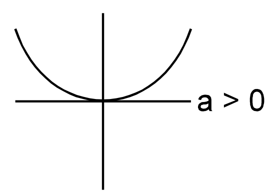
  E.g.: $y^2 = 4ax$.



- **Symmetry about y-axis :**

  If all the exponents of 'x' in the equation are even, then the curve (graph) displays symmetry about the y-axis.

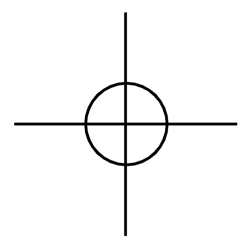
  E.g.: $x^2 = 4ay$.



- **Symmetry about both axis :**

  If all the exponents of both 'x' and 'y' in the equation are even, then the curve (graph) exhibits symmetry about both the x-axis and y-axis.

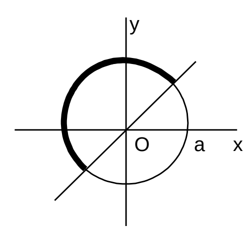
  E.g.: $x^2 + y^2 = a^2$.



- **Symmetry about the line y = x :**

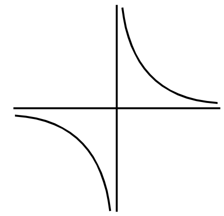
  If the equation of the curve remains unaltered upon swapping 'x' and 'y', then the curve (graph) demonstrates symmetry about the line $y = x$.

  E.g. : $x^2 + y^2 = a^2$



- **Symmetry in opposite quadrants :**

  If the equation of the curve (graph) remains unchanged when 'x' and 'y' are substituted with '-x' and '-y' respectively, then symmetry exists across opposite quadrants.

  E.g.: $xy = c^2$

(b) Determine the points of intersection between the curve and the x-axis as well as the y-axis.

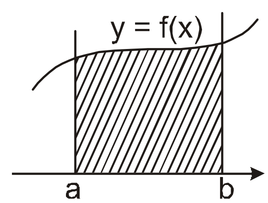
(c) Calculate $\frac{dy}{dx}$ and set it equal to zero to identify the locations on the curve where horizontal tangents exist.

(d) Analyze the intervals where f(x) experiences growth or decline.

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$

(ii) Area included between the curve y = f(x), x-axis and the ordinates

$$x = a, x = b$$



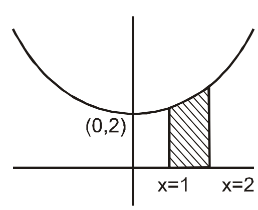
Graph of y = f(x)

(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve

$$y = f(x), \text{x-axis}, x = a \text{ and } x = b \text{ is } \int_a^b f(x)dx$$

Ex.1 Determine the area bounded by the curve $y = x^2 + 2$, the x-axis, and the vertical lines x = 1 and x = 2.

Sol. Graph of $y = x^2 + 2$



$$\text{Area} = \int_1^2 (x^2 + 2)dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_1^2 = \frac{13}{3}$$

Ex.2 Determine the area enclosed by the curve $y = \ln x + \tan^{-1} x$ and x-axis between ordinates x = 1 and x = 2.

Sol.
$$y = \ln x + \tan^{-1}x$$
Domain $x > 0$,
$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$$

y is increasing and $x = 1$, $y = \frac{\pi}{4}$

y is positive in [1, 2]

$$\text{Required area} = \int_1^2 \left(\ell n x + \tan^{-1} x\right) dx$$

$$= \left[x \ell n x - x + x\tan^{-1}x - \frac{1}{2}\ell n\left(1+x^2\right)\right]_1^2$$

$$= 2\ln 2 - 2 + 2\tan^{-1}2 - \frac{1}{2}\ln 5 - 0 + 1 - \tan^{-1}1 + \frac{1}{2}\ln 2$$

$$= \frac{5}{2}\ln 2 - \frac{1}{2}\ln 5 + 2\tan^{-1}2 - \frac{\pi}{4} - 1$$

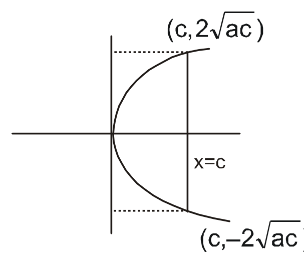
Note: If a function is confirmed to have positive values, a graph may not be required.

Ex.3 The value of k is determined by the fact that the area enclosed by any double ordinate on a parabola is k times the area of the rectangle formed by the double ordinate and its distance from the vertex.

Sol. Consider $y^2 = 4ax$, $a > 0$ and $x = c$

Area by double ordinate $= 2\int_0^c 2\sqrt{a}\sqrt{x}dx$

$$= \frac{8}{3}\sqrt{a}c^{\frac{3}{2}}$$



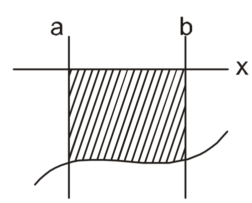
Figure

Area by double ordinate = k (Area of rectangle)

$$\frac{8}{3}\sqrt{a}c^{\frac{3}{2}} = k4\sqrt{a}c^{\frac{3}{2}}$$

$$k = \frac{2}{3}$$

(b) If $f(x) < 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $-\int_a^b f(x)dx$



Graph of y = f(x)

Ex.4 What is the area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between x = 1 and x = 2

Sol. A rough graph of $y = \log_{\frac{1}{2}} x$ is as follows

$$\text{Area} = -\int_1^2 \log_{\frac{1}{2}} x dx$$

$$= -\int_1^2 \log_e x . \log_{\frac{1}{2}} e dx$$

$$= -\log_{\frac{1}{2}} e . \left[x \log_e x - x\right]_1^2$$

$$-\log_{\frac{1}{2}} e \cdot (2\log_e 2 - 2 - 0 + 1)$$

$$-\log_{\frac{1}{2}} e \cdot (2\log_e 2 - 1)$$

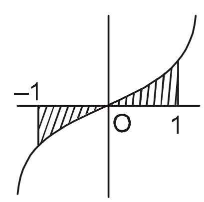
Note: If y = f(x) does not change sign in [a, b], then area bounded by y = f(x), x-axis between ordinates x = a, x = b is $\left| \int_a^b f(x) \, dx \right|$

(c) If $f(x) \geq 0$ for $x \in [a, c]$ and $f(x) \leq 0$ for $x \in [c, b]$ $(a < c < b)$ then area bounded by curve y = f(x) and x-axis between x = a and x = b is

$$\int_a^c f(x) dx - \int_c^b f(x) dx$$

Ex.5 Find the area bounded by $y = x^3$ and x- axis between ordinates x = –1 and x = 1

Sol. Required area $= \int_{-1}^{0} -x^3 dx + \int_0^1 x^3 dx$



Graph of y = x^3

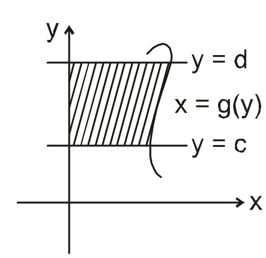
$$= \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_0^1$$

$$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0$$

$$= \frac{1}{2}$$

Note: Most general formula for area bounded by curve y = f(x) and x- axis between ordinates x = a and x = b is

$$\int_a^b |f(x)| dx$$

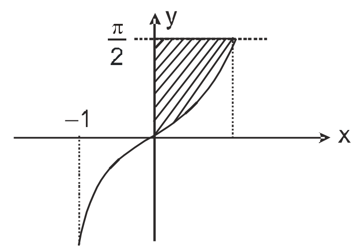


Graph of x = g(y)

Ex.6 Find the area enclosed between the curve $y = \sin^{-1}x$ and the y-axis in the interval $y = 0$ and $y = \frac{\pi}{2}$

Sol.

$$y = \sin^{-1} x$$
$$x = \sin y$$
$$\text{Required area} = \int_0^{\frac{\pi}{2}} \sin y\, dy$$
$$= -\cos y \Big]_0^{\frac{\pi}{2}}$$
$$= -(0-1) = 1$$



Graph of $y = \sin^{-1}x$

Note: The area in the above example can also be calculated through integration with respect to x.

Area = (area of rectangle formed by $x = 0, y = 0, x = 1, y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between $x = 0$ and $x = 1$)

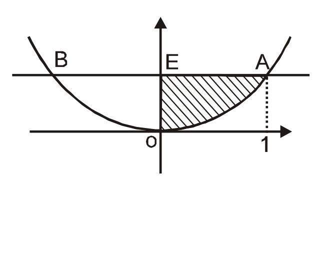
$$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1} x\, dx$$
$$= \frac{\pi}{2} - \left(x \sin^{-1} x + \sqrt{1-x^2}\right)_0^1$$
$$= \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1\right) = 1$$

Ex.7 Determine the area enclosed by the parabola $x^2 = y$, the y-axis, and the line $y = 1$.

Sol. Graph of $y = x^2$

Area OEBO = Area OAEO

$$= \int_0^1 |x|\, dy = \int_0^1 \sqrt{y}\, dy$$
$$= \frac{2}{3}$$

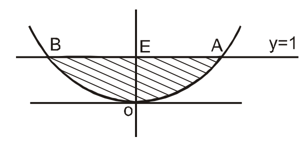


Ex.8 Find the area bounded by the parabola $x^2 = y$ and line $y = 1$.

Sol. Graph of $y = x^2$

Required area is area OABO = 2 area (OAEO)

$$= 2\int_0^1 |x| dy = 2\int_0^1 \sqrt{y} dy$$



$$= \frac{4}{3}$$

Note: General formula for the area enclosed by the curve $x = g(y)$ and y–axis between abscissa

$$y = c \text{ and } y = d \text{ is } \int_{y=c}^{d} |g(y)| dy$$