INTEGRALS

PROPERTIES OF DEFINITE INTEGRALS

Basic Properties of Definite Integral

Property 1 $\int_a^b f(x)dx = \int_a^b f(z)dz$

Property 2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Property 3 $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^{b} f(x)dx$

Where $a < c_1 < c_2 < \ldots\ldots\ldots < c_{n-1} < c_n < b$

Ex.1 $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ \sqrt{x} & 1 \le x \le 2 \end{cases}$ find $\int_0^2 f(x)dx$.

Sol. We have $\int_0^2 f(x)dx - \int_0^1 f(x)dx + \int_0^2 f(x)dx$

$$= \int_0^1 x^2 dx + \int_1^2 \sqrt{x} dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^{3/2}}{3/2}\right]_1^2$$

$$= \frac{1}{3} + \frac{2^{3/2}}{3/2} - \frac{1}{3/2} - \frac{2}{3}(2\sqrt{2}) - \frac{1}{3} - \frac{1}{3}[4\sqrt{2} - 1]$$

Property 4 $\int_0^a f(x)dx = \int_0^d f(a-x)dx$

Ex.2 Evaluate $\int_0^{\pi/4} \log(1+\tan x)dx$

Sol.

$$I - \int_0^{\pi/4} \log(1+\tan x)dx$$

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$I = \int_0^{\frac{\pi}{4}} \log\{\log 2 - \log(1+\tan x)\}dx$$

$$I = \log 2\int_0^{\pi/4} dx - I$$

$$2J = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2.$$

Property 5 $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(-x) = f(x) \end{cases}$

Ex.3 Evaluate

(i) $\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right)dx$ (ii) $\int_{-\pi}^{\pi} \sin mx \cos nx dx$

Sol. In (i) $\log\frac{1+x}{1-x}$ is an odd function, then integrand is an odd function and in (ii) sin (mx) is an odd function so integration is an odd function.

(i) $I = 0$ (ii) $I = 0$

Property 6 $\int_0^{2a} f(x)dx - \int_0^{2} f(x)dx + \int_0^{2} f(2a-x)dx$

Proof: $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{2a} f(x)dx$

In second integral of RHS substitute $x = 2a - y, dx = -dy$

Then, $\int_0^{2a} f(x)dx$

$$= \int_0^{a} f(x)dx - \int_a^{0} f(2a-y)dy$$

$$= \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

Property 7 In particular, $\int_0^{2a} f(x)dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^{a} f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$

Property 8 In Particular $\int_a^{a+n/} f(x)dx = n\int_0^{1} f(x)dx; n \in I$

$$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^{T} f(x)dx; m, n \in I$$