INTEGRALS

INTEGRATION BY PARTIAL FRACTIONS

Integration of rational algebraic functions by using partial fractions:

(i) Partial Fractions:

If f(x) and g(x) represent two polynomials, the expression $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x. Assume the degree of f(x) is less than the degree of g(x); if not, perform division of f(x) by g(x) until the degree of the numerator is lower than that of the denominator. Then, apply the concept of partial fractions as outlined below:

CASE I :

When the denominator can be expressed as the product of non-repeating linear factors.

Let's assume... $g(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$.

Then, we assume that $\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots\ldots+\frac{A_n}{x-a_n}$

Where A_1, A_2, An are constants that can be determined by equating the numerator on the right-hand side to the numerator on the left-hand side and subsequently substituting them.

$$x=a_1, a_2, \ldots\ldots\ldots, a_n.$$

CASE II:

When the denominator g(x) can be represented as the product of linear factors, with some of them being repeated.

Example:

$$\frac{1}{g(x)}=\frac{1}{(x-a)^k(x-a_1)(x-a_2)\ldots\ldots(x-a_r)}$$

This can be expressed as

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+.....+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+....+\frac{B_r}{(x-a_r)}$$

To find the constants, we equate the numerators on both sides. Some of these constants are determined through substitution, as in Case I, while the remaining constants are obtained by equating the coefficients of the same power of x. The procedure is illustrated in the following example.

CASE III:

When certain factors of the denominator g(x) are quadratic but non-repeating, for each quadratic factor $ax^2 + bx + c$, we posit a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.

Here, A and B are constants determined by comparing coefficients of corresponding powers of x in the numerators on both sides. In practice, it is recommended to assume partial fractions of this form $\frac{A(2ax+b)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}$. The following example illustrates the procedure.

CASE IV :

When certain factors of the denominator g(x) are quadratic and recurring, fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c}+\frac{A_1}{ax^2+bx+c}\right\}+\left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2}+\frac{A_2}{(ax^2+bx+c)^2}\right\}+.....\left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k}+\frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$$

Ex.1 Evaluate $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Sol. Let $\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$

$\Rightarrow \frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$

Putting $x = 1, \ -6A = 1$

$\Rightarrow$ $A = -\frac{1}{6}$

Putting $x = 3, \ 10C = 5$

$\Rightarrow$ $C = \frac{1}{2}$

Putting $x = -2, 15B = 5$

$\Rightarrow$ $B = -\frac{1}{3}$

So $-\frac{1}{6}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{2}\int \frac{1}{x-3}dx$

$$= -\frac{1}{6} \log|x-1| - \frac{1}{3} \log_e|x+2| + \frac{1}{2} \log_e|x-3| + C$$

Ex.2 Solve $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$ into partial fractions.

Sol. In this case, the provided function is an improper rational function, meaning that the degree of the numerator is greater than the degree of the denominator.

On dividing,

We get, $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = x - 1 + \frac{(-x+4)}{(x^2 - 5x + 6)}$ .... (i)

We have, $\frac{-x+4}{x^2 - 5x + 6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

Then $-x + 4 = A(x-3) + B(x-2)$ .... (ii)

Putting $x - 3 = 0$ or $x = 3$ in (ii),

We get $1 = B(1)$

$\Rightarrow$ $B = 1.$

Putting $x - 2 = 0$ or $x = 2$ in (ii),

We get $2 = A(2-3)$

$\Rightarrow \quad A = -2$

$\therefore \quad \dfrac{-x+4}{(x-2)(x-3)} = \dfrac{-2}{x-2} + \dfrac{1}{x-3}$

Hence $\quad \dfrac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 - \dfrac{2}{x-2} + \dfrac{1}{x-3}$

Ex.3 Resolve $\dfrac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate

$\int \dfrac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$.

Sol. Let $\quad \dfrac{3x-2}{(x-1)^2(x+1)(x+2)} = \dfrac{A_1}{x-1} + \dfrac{A_2}{(x-1)^2} + \dfrac{A_3}{x+1} + \dfrac{A_4}{x+2}$

$\Rightarrow \quad 3x - 2$

$A_1(x-1)(x+1)(x+2) + A_2(x+1)(x+2) + A_3(x-1)^2(x+2) + A_4(x-1)^2(x+1)$ ......(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i)

we get $\quad 1 = A_2 (1 + 1) (1 + 2)$

$\Rightarrow \quad A_2 = -\dfrac{5}{4}$

Putting $x + 1 = 0$ or, $x = -1$ in (i)

we get $\quad -5 = A_3 (-2)^2 (-1 + 2)$

$\Rightarrow \quad A_3 = -\dfrac{5}{4}$

Putting $x + 2 = 0$ or, $x = -2$ in (i)

we get $\quad -8 = A_4 (-3)^2 (-1)$

$\Rightarrow \quad A_4 = \dfrac{8}{9}$

Now equating coefficient of x^3 on both sides,

we get $\quad 0 = A_1 + A_3 + A_4$

$\Rightarrow \quad A_1 = -A_3 - A_4$

$$\frac{5}{4} - \frac{8}{9} = \frac{13}{36}$$

$$\frac{3x-2}{(x-1)^2(x+1)(x+2)}=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

Hence $$\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$$

$$\frac{13}{36}\text{In}|x-1|-\frac{1}{6(x-1)}-\frac{5}{4}\text{In}|x+1|+\frac{8}{9}\text{In}|x+2|+C$$

Ex.4 Evaluate $\int\frac{x^2}{(x^2+4)(x^2+1)}dx$

Sol.
$$\int\frac{x^2}{(x^2+4)(x^2+1)}dx$$

$$=\frac{1}{3}\int\left[\frac{4}{x^2+4}-\frac{1}{x^2+1}\right]dx$$

$$=\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}x+C$$

$$=\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{3}\tan^{-1}x+C$$

Ex.5 Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\frac{2x-3}{(x-1)(x^2+1)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$. Then,

$$2x-3=A(x^2+1)^2+(Bx+C)(x-1)(x^2+1)+(Dx+E)(x-1)\ \ldots\ldots(i)$$

Putting $x=1$ in eq (i), we get $-1=A(1+1)^2\Rightarrow A=-\frac{1}{4}$

Comparing coefficients of like powers of x on both side of (i), we have

$$A+B=0, C-B=0, 2A+B-C+D=0, C+E-B-D=2 \text{ and}$$

$$A-C-E=-3.$$

Putting $A=-\frac{1}{4}$ and solving these equations, we get

$$B=\frac{1}{4}=C, D=\frac{1}{4}\text{ and}$$

$$E = \frac{5}{2} \therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{9x+5}{2(x^2+1)^2}$$

Ex.6 Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

Then, $2x = A(x^2+x+1) + (Bx+C)(x-1)$ ....(i)

Putting x – 1 = 0 or, x = 1 in (i),

we get $2 = 3A$

$\Rightarrow$ $A = \frac{2}{3}$

Putting x = 0 in (i),

we get $A - C = 0$

$\Rightarrow$ $C = A = \frac{2}{3}$

Putting x = – 1 in (i),

we get $-2 = A + 2B - 2C.$

$\Rightarrow$ $-2 = \frac{2}{3} + 2B - \frac{4}{3}$

$\Rightarrow$ $B = -\frac{2}{3}$

$\therefore$ $\frac{2x}{x^3-1} = \frac{2}{3}.\frac{1}{x-1} + \frac{\left(-\frac{2}{3}\right)x + \frac{2}{3}}{x^2+x+1}$

Or $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$