INTEGRALS

INTEGRALS BY SUBSTITUTION

Evaluation of Definite Integral by change of variable

When the variable x in a definite integral is replaced with t, the substitution impacts three locations.

(1) The expression inside the integral is modified.

(2) dx is replaced with df(t).

(3) The upper and lower limits are modified.

Ex.1 Evaluate $\int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx$

Sol. Substitute $x = \tan\theta$, $dx = \sec^2\theta d\theta$

(limits of integration are changed to 0 and $\frac{\pi}{2}$)

$$I = \int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta)\sec^2\theta} \sec^2\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{1+\tan\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\theta\left(\frac{\pi}{2}-\theta\right)} d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta} d\theta$$

Adding (1) and (2),

We get, $$2l = \int_0^{\pi/2} d\theta = \pi / 2$$

$$I = \frac{\pi}{4}$$

Some more properties of Definite Integral

Property 9 If $f(x) \geq 0 \forall x \in [a,b]$, then $\int_a^b f(x)dx \geq 0$

Property 10 If $f(x)$ is an odd function of x, then $\int_0^{\hat{f}} f(t)dt$ is an even function of x.

Ex.2 Show that if f(t) is an odd function then $F(x) = \int_a^x f(t)dt$ is an even function.

Sol. We have $$F(x) = \int_a^0 f(t)dt + \int_0^x f(t)dt$$

$$F(-x) = \int_a^0 f(t)dt + \int_0^{-x} f(t)dt$$

Substituting $t = -u$ in the 2^{nd} integral

$$F(-x) = \int_a^0 f(t)dt + \int_0^x f(-u)(-du)$$

$$= \int_a^0 f(t)dt + \int_0^x f(u)(du) \quad (\therefore f(-u) = -f(u))$$

$$= \int_a^x f(t)dt + F(x)$$

Therefore F(x) is an even function