INTEGRALS

INTEGRALS OF SOME PARTICULAR FUNCTIONS

Integration of type

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$$

Express ax^2+bx+c as a perfect square and then utilize the standard results.

Ex.1 Evaluate: $\int \sqrt{x^2+2x+5}\,dx$

Sol. We have,

$$\int \sqrt{x^2+2x+5}$$

$$\int \sqrt{x^2+2x+1+4}\,dx$$

$$\int \sqrt{(x+1)^2+2^2}$$

$$\frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2}+\frac{1}{2}\cdot(2)^2 \text{In}\left|(x+1)+\sqrt{(x+1)^2+2^2}\right|+C$$

$$\frac{1}{2}(x+1)\sqrt{x^2+2x+5}+2\text{In}\left|(x+1)+\sqrt{x^2+2x+5}\right|+C$$

Ex.2 Evaluate: $\int \frac{1}{x^2-2x+3}dx$

Sol.

$$I=\int \frac{1}{x^2-2x+3}dx$$

$$\int \frac{1}{(x-1)^2+2}dx$$

$$\int \frac{1}{(x-1)^2+(\sqrt{2})^2}dx$$

$$\int \frac{1}{(x-1)^2+(\sqrt{2})^2}dx$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$$

Ex.3 Evaluate: $\int \frac{1}{\sqrt{33+8x-x^2}}\ dx$

Sol.

$$\int \frac{1}{\sqrt{33+8x-x^2}}dx$$

$$\int \frac{1}{\sqrt{-\left\{x^2-8x-33\right\}}}dx$$

$$\int \frac{1}{\sqrt{-\left\{x^2-8x+16-49\right\}}}dx$$

$$\int \frac{1}{\sqrt{-\left\{(x-4)^2-7^2\right\}}}dx$$

$$\int \frac{1}{\sqrt{7^2-(x-4)^2}}dx$$

$$\sin^{-1}\left(\frac{x-4}{7}\right)+C$$

Integration of type

$$\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int (px+q)\sqrt{ax^2+bx+c}dx$$

Represent px + q as A times the derivative of the denominator plus B.

Ex.4 Evaluate: $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\ dx$

Sol.

$$\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx$$

$$\int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx \triangleleft$$

$$\int \frac{2x+4}{\sqrt{x^2+4x+1}}dx - \int \frac{1}{\sqrt{x^2+4x+1}}dx$$

$$\int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}dx$$

$$t = \left(x^2+4x+1\right)$$ (For I^{ST} integral)

$$2\sqrt{t}-\text{In}|(x+2)+|+\sqrt{x^2+4x+1}C$$

$$2\sqrt{x^2+4x+1}-\text{In}\left|x+2+\sqrt{x^2+4x+1}\right|+C$$

Ex.5 Evaluate: $\int (x-5)\sqrt{x^2+x}\,dx$

Sol.

$$(x-5)=\text{In}\cdot\frac{d}{dx}\left(x^2+x\right)+\mu.$$

$$x-5=\lambda(2x+1)+\mu$$

By comparing coefficients of similar powers of x, we obtain

$$1=2\lambda \text{ and } \lambda+\mu=-5$$

$$\lambda=\frac{1}{2} \text{ and } \mu=-\frac{11}{2}$$

$$\int (x-5)\sqrt{x^2+x}dx$$

$$\int\left(\frac{1}{2}(2x+1)-\frac{11}{2}\right)\sqrt{x^2+x}dx$$

$$\int\frac{1}{2}(2x+1)\sqrt{x^2+x}dx-\frac{11}{2}\int\sqrt{x^2+x}dx$$

$$\frac{1}{2}\int\sqrt{t}dt-\frac{11}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

(Where $t=x^2+x$ for first integral)

$$\frac{1}{2}\cdot\frac{t^{3/2}}{3/2}-\frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]\right.$$

$$\left.-\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2\text{In}\left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}\right]+C$$

$$=\frac{1}{3}t^{3/2}-\frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right]+C$$

$$=\frac{1}{3}\left(x^2+x\right)^{3/2}-\frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right]+C$$