INTEGRALS

INTRODUCTION & INTEGRATION AS AN INVERSE PROCESS OF DIFFERENTIATION

If f and g are functions of x such that $g'(x) = f(x)$, then the indefinite integration of $f(x)$ with respect to x is defined and represented as $\int f(x)\,dx = g(x) + C$, where C is referred to as the **constant of integration.**

Standard formulae:

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$

2. $\int \frac{dx}{ax+b} = \frac{1}{a} \text{In} | ax+b | + C$

3. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

4. $\int a^{px+q} dx = \frac{1}{P} \frac{a^{px+q}}{\ell na} + C; a > 0$

5. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

6. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

7. $\int \tan(ax+b)dx = \frac{1}{a} \text{In} | \sec(ax+b) | + C$

8. $\int \cot(ax+b)dx = \frac{1}{a} \text{In} | \sin(ax+b) | + C$

9. $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$

10. $\int \text{cos ec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$

11. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

12. $\int \cos ec(ax+b).\cot(ax+b)dx = -\frac{1}{a}\cos ec(ax+b)+C$

13. $\int \sec x dx = \text{In}\,|\sec x + \tan x| + C$

OR

$$\text{In}\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$$

14. $\int \cos ecx dx = \lambda n\,|\cos ecx - \cot x| + C$

OR

$$\text{In}\left|\tan\frac{x}{2}\right|+C$$

OR

$$-\text{In}\,|\cos ex + \cot x| + C$$

15. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$

16. $\int \frac{dx}{a^2-x^2} = \frac{x}{a}\tan^{-1}\frac{x}{a}+C$

17. $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$

18. $\int \frac{dx}{\sqrt{x^2+a^2}} = \text{In}\,|x+\sqrt{x^2+a^2}\,| + C$

OR

$$\sinh^{-1}\frac{x}{a}+C$$

19. $\int \frac{dx}{\sqrt{x^2-a^2}} = \text{In}\,|x+\sqrt{x^2-a^2}\,| + C$

OR

$$\cosh^{-1}\frac{x}{a}+C$$

20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \text{In}\left|\frac{a+x}{a-x}\right| + C$

21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \text{In}\left|\frac{x-a}{x+a}\right| + C$

22. $\int \sqrt{a^2 - x^2}dx = \frac{a^2}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-x} + C$

23. $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\text{In}\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C$

24. $\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\text{In}\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

25. $\int e^{ax} \cdot \sin bxdx = \frac{e^{ax}}{a^2 + h^2}(a\sin bx - b\cos bx) + c$

26. $\int e^{ax} \cdot \cos bxdx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$

Theorems on integration:

1. $\int Cf(x) \cdot dx = C\int f(x)dx$

2. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

3. $\int f(x)dx = g(x) + C_1$

$$\int f\left(ax+b\right)dx = \frac{g\left(ax+b\right)}{a} + C_2$$

Ex.1 Solve: $\int 4x^5 dx$

Sol.

$$\int 4x^5 dx$$

$$\frac{4}{6}x^6 + C$$

$$\frac{2}{3}x^6 + C$$

Ex. 2 Evaluate: $\int\left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right)dx$

Sol.

$$\int\left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right)dx$$

$$=\int x^3dx+\int 5x^2dx-\int 4dx+\int\frac{7}{x}dx+\int\frac{2}{\sqrt{x}}dx$$

$$=\int x^3dx+5.\int x^2dx-4.\int 1dx+7.\int\frac{1}{x}dx+2\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^4}{4}+5.\frac{x^3}{3}-4x+7\text{In}\,|x|+2\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+C$$

$$=\frac{x^4}{4}+\frac{5}{3}x^3-4x+7\text{In}\,|x|+4\sqrt{x}+C$$

Ex.3 Evaluate: $\int\left(e^{2\ell nx}+e^{a\ell nx}+e^{4\ell nx}\right)dx, a>0$

Sol.

$$\int\left(e^{2\ell nx}+e^{a\ell nx}+e^{4\ell nx}\right)dx$$

$$=\int\left(e^{\ell nx^2}+e^{\ell nx^a}+e^{\ell nx^4}\right)dx$$

$$=\int\left(x^2+x^3+x^4\right)dx$$

$$=\frac{x^3}{3}+\frac{x^{3+1}}{a+1}+\frac{x^5}{5}+c$$

Ex. 4 Evaluate: $\int\left(\frac{2^{x+1}-5^{x-1}}{10^x}\right)dx$

Sol.

$$\int\left(\frac{2^{x+1}-5^{x-1}}{10^x}\right)dx$$

$$=\int\left[2\left(\frac{1}{5}\right)^x-\frac{1}{5}\left(\frac{1}{2}\right)^x\right]dx$$

$$= \frac{2\left(\frac{1}{5}\right)^x}{\log_e\left(\frac{1}{5}\right)} - \frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\log\left(\frac{1}{2}\right)} + C$$

Ex. 5 Evaluate: $\int \sec^2 x \operatorname{cosec}^2 x \, dx$

Sol.

$$I = \int \sec^2 x \operatorname{cosec}^2 x$$

$$\int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x}$$

$$\int \left(\sec^2 x + \operatorname{cosec}^2 x\right) dx$$

$$\tan x - \cot x + C$$

Ex. 6 Evaluate: $\int \frac{(1+x)^3}{\sqrt{x}} \, dx$

Sol.

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

$$\int \frac{1+3x+3x^2+x^3}{\sqrt{x}} dx$$

$$\int x^{-\frac{1}{2}} + 3\int x^{\frac{1}{2}} dx + 3\int x^{\frac{3}{2}} dx + \int x^{\frac{5}{2}} dx$$

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + C$$

$$2\sqrt{x} + 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + C$$

Ex. 7 Evaluate: $\int \frac{1}{4+9x^2} \, dx$

Sol. We have

$$\int \frac{1}{4+9x^2} dx$$

$$\frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2} dx$$

$$\frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2 + x^2} dx$$

$$\frac{1}{9} \cdot \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \frac{x}{\left(\frac{2}{3}\right)} + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{3x}{2}\right) + C$$

Ex. 8 Evaluate: $\int \cos x \cos 2x \, dx$

Sol.

$$\int \cos x \cos 2x dx$$

$$\frac{1}{2}\int 2\cos x \cos 2x dx$$

$$\frac{1}{2}\int (\cos 3x + \cos x) dx$$

$$\frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C$$