INTEGRALS

DEFINITE INTEGRAL

In this context, we will define integration as the process of summation or the definite integral as the limit of a sum. Subsequently, we will explore some properties of the definite integral. The concept of the definite integral is then applied to determine the area enclosed by specific curves.

DEFINITE INTEGRAL

Consider a continuous real-valued function, denoted as f(x), defined on the interval [a, b], where:

Then $\int f(x)dx = F(x) + C.$

Then $\int_a^b f(x)dx = F(b) - F(a)$ called definite integral of f(x) in [a, b].

Remark:

1. If f(x) is discontinuous at x = a but continuous at x = b

   Then $\int_a^b f(x)dx = F(b) - \lim_{x\to a+} F(x)$

2. If f(x) is discontinuous at x = b but continuous at x = a

   Then $\int_a^b f(x)dx = \lim_{x\to b-} F(x) - F(a)$

3. If f(x) exhibits discontinuity at both x = a and x = b

   Then $\int_a^b f(x)dx = \lim_{e\to 0-} \int_{a+e}^{b-e} f(x)dx$ or $\lim_{x\to b-} F(x) - \lim_{x\to a+} F(x)$

4. If f(x) experiences discontinuity at x = c, where a < c < b

   Then $\int_a^b f(x)dx = \lim_{e\to 0} \int_a^{c-e} f(x)dx + \lim_{e\to 0} \int_{c+e}^{b} f(x)dx$

Note: Even if f(x) is not defined at x = a, x = b, or at both, the integral $\int_a^b f(x)dx$ can still be evaluated. Here, a and b are referred to as the lower and upper limits of integration, respectively. In the case of a change in variable (i.e., substitution), the limits of integration should be adjusted accordingly.

Ex.1 Evaluate $\int_0^4 \left(x - 2\sqrt{x} + x^2\right)dx$

Sol.

$$\int_0^4 \left(x - 2\sqrt{x} + x^2\right)dx - \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} + \frac{x^3}{3}\right]_0^4$$

$$-\left(8 - \frac{32}{3} + \frac{64}{3}\right) - (0)$$

$$= \frac{56}{3}$$

Definite Integral as the Limit of a Sum

(i) Represent the provided series in the format of $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) The sum of the series as n approaches infinity is the limit. $\lim_{n\to 0} \sum_{r=0}^{n-1} \frac{1}{n} \cdot f\left(\frac{r}{n}\right)$ Replace r/n by x, 1/n by dx and $\lim_{n\to\infty} \sum$ by the sign of integration $\int$

(iii) The lower and upper integration bounds correspond to the values of r/n for the initial and final terms (or the limits of these values, respectively).

Some Important Formulae

1. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

2. $\sum_{n=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{n=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

4. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta) - \frac{\sin\frac{2\beta}{2}}{\sin\frac{\beta}{2}} \sin\left[\frac{1^{st}\ \text{angle} + \text{lastangle}}{2}\right]$

5. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\sin n\beta/2}{\sin\beta/2}\cos\left[\frac{1^{st}\ \text{angle} + \ \text{last angle}}{2}\right].$$

6. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots \ \text{to}\,00 = \log_6 2$

7. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2}\ldots\ldots\ldots = \frac{\pi^2}{12}.$

8. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{6}$

9. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\cdots = \frac{\pi^2}{8}$

10. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{24}.$

11. $\cos\theta = \frac{e^{18} + e^{-18}}{2} \cdot \sin\theta = \frac{e^{B} - e^{-18}}{2}$

12. $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$

Ex.2 Evaluate the following limits (utilizing definite integrals).

(1) $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \ldots + \sqrt{1+\frac{n}{n}}\right)$

(2) $\lim_{n\to\infty} \frac{\pi}{2n}\left(1 + \cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \ldots + \cos\frac{(n-1)\pi}{2n}\right)$

Sol. (1)

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \ldots + \sqrt{1+\frac{n}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\sqrt{1+\frac{r}{n}}$$

$$= \int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{\frac{3}{2}}\right]_0^1 = \frac{2^{3/2}}{\frac{3}{2}} - \frac{1}{3} - \frac{2}{3}\{2\sqrt{2} - 1\}$$

(2) $$\sum_{k=0}^{n-1}\cos(a+kd)=\frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}.\cos\left(a+\frac{k-1d}{2}\right)$$

$$\sum_{k=0}^{n-1}\cos(a+kd)=\cos(0\bullet d)+\cos(1\bullet d)+\cos(2\bullet d)+\ldots=1+\cos(1\bullet d)$$

$$\text{Where, } a=0, d=\frac{\pi}{2n}$$

$$\lim_{n\to\infty}\frac{\pi}{2n}\left[\frac{\sin\left(\frac{n\bullet\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}\cos\left(0+\frac{(n-1)\pi}{4n}\right)\right]$$

$$\lim_{n\to\infty}\frac{\pi}{2n}\bullet\frac{1}{\sqrt{2}}\bullet\frac{1}{\frac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}\bullet\left(\frac{\pi}{4n}\right)}$$

$$\left[\cos\frac{(n-1)\pi}{4n}\right]\left\{\begin{array}{l}\lim_{x\to0}\frac{\sin x}{x}=1\\ I_F n\to\infty\Rightarrow\frac{1}{n}\to0\end{array}\right\}$$

$$\lim_{n\to\infty}\frac{\pi}{2n}\frac{1}{\sqrt{2}}\frac{1}{1\bullet\left(\frac{\pi}{4n}\right)}\left[\cos\pi\left(\frac{1}{4}-\frac{1}{4n}\right)\right]$$

$$\lim_{n\to\infty}\frac{\pi}{2\sqrt{2}}\times\frac{4}{\pi}\left[\cos\pi\left(\frac{1}{4}-\frac{1}{4n}\right)\right]$$

$$\frac{4}{2\sqrt{2}}\cos\left(\frac{\pi}{4}\right)$$

$$\frac{4}{2\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{4}{4}=1.$$