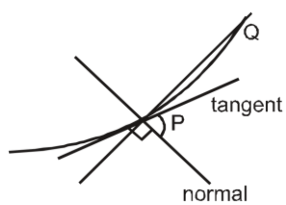
APPLICATIONS OF DERIVATIVES

TANGENTS & NORMALS (FOR COMPETITIVE EXAM)

Tangent and Normal:

Consider the function y = f (x) with the graph depicted in the figure. Take the secant PQ, and as point Q approach point P along the curve, denoted as Q1, Q2… i.e. $Q \to P$, the secant PQ transforms into the tangent at point P. the line passing through P and perpendicular to the tangent is referred to as the normal at P.



Figure

Geometrical Meaning of $\frac{dy}{dx}$:

As $Q \to P$, $h \to 0$, and the slope of chord PQ approaches the slope of the tangent at P (refer to the figure.)

$$PQ = \frac{f(x+h)-f(x)}{h}$$

$\lim\limits_{Q\to P}$ Slope of chord $PQ = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Slope of tangent at $P = f'(x) = \frac{dy}{dx}$

Equation of tangent and normal:

$\left.\frac{dy}{dx}\right]_{(x_1,\, y_1)} = f'(x_1)$ represent the slope of the tangent at the point (x_1, y_1) on the curve y = f (x). Therefore, the equation of the tangent at (x_1, y_1) is expressed as

$(y - y_1) = f(x_1)(x - x_1)$ when $f(x_1)$ is a real number.

Moreover, considering that the normal is a line perpendicular to the tangent (x_1, y_1), its equation is expressed as $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$ when f (x_1) is a nonzero real number.

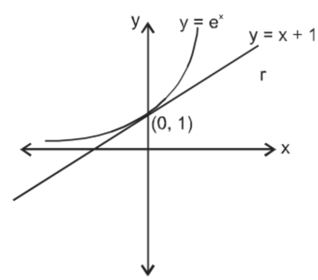
If $f'(x_1) = 0$, then the tangent is represent by the equation $y = y_1$. And the normal is represent by the equation $x = x_1$.

If $$\lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h} = \infty \text{ or } -\infty,$$

Then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Ex1. Determine the equation of the tangent to $y = e^x$ at $x = 0$. Subsequently, sketch the graph.

Sol. At $x = 0$



$\Rightarrow$ $y = e^0 = 1$

$\frac{dy}{dx} = e^x$

$\Rightarrow$ $\left.\frac{dy}{dx}\right|_{x=0} = 1$

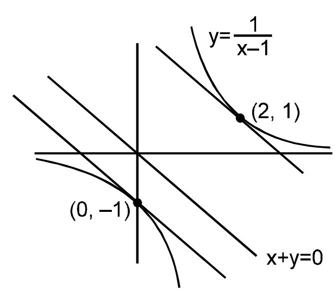
Hence equation of tangent is

$1(x - 0) = y - 1)$

$\Rightarrow$ $y = x + 1$

Ex.2 Determine the equation of all straight lines that are tangent to the curve $y = \frac{1}{x-1}$ and are parallel to the line $x + y = 0$.

Sol. Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow$ $\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$

$\Rightarrow$ $-\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow$ $x_1 = 0$ or 2

$\Rightarrow$ $y_1 = -1$ or 1

Hence, tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$$-1(x - 0) = (y + 1)$$

and $$-1(x - 2) = (y - 1)$$

$\Rightarrow$ $$x + y + 1 = 0$$

and $$y + x = 3$$

Ex.3 Determiner the equation of the normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Sol. In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence, the point of contact is (– 2, 2)

$$\frac{dy}{dx} = 2x + 1$$

$\Rightarrow$ $$\left.\frac{dy}{dx}\right|_{x=-2} = -3.$$

So, the slope of normal at (– 2, 2) is .

Hence, equation of normal is

$$\frac{1}{3}(x + 2) = y - 2$$

$\Rightarrow$ $$3y = x + 8$$

Ex.4 Determine the equations of the tangent and normal to the curve. $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$ at the point $t = \frac{1}{2}$.

Sol. Given that $x = \frac{2at^2}{1+t^2}$ $\quad y = \frac{2at^3}{1+t^2}$

At $t = \frac{1}{2}$, $x = \frac{2a}{5}$, $y = \frac{a}{5}$

Also $$\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$$

And $$\frac{dy}{dt} = \frac{2at^2(3+t^2)}{(1+t^2)^2}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2}\,t\,(3 + t^2)$$

When $\quad t = \frac{1}{2}, \frac{dy}{dx} = \frac{1}{2}\frac{1}{2}\left(3+\frac{1}{4}\right) = \frac{13}{16}$

$\therefore$ The equation of the tangent when $t = \frac{1}{2}$ is

$$y - \frac{a}{5} = \left(\frac{13}{16}\right)\left(x-\frac{2a}{5}\right)$$

$\Rightarrow \qquad 13x - 16y = 2a$

and the equation of the normal is $\left(y-\frac{a}{5}\right)\left(\frac{13}{16}\right) + x - \frac{2a}{5} = 0$

$\Rightarrow \qquad 16x + 13y = 9a$

Ex.5 Determine the value of 'c' such that the line joining the points (0, 4) and (5, –1) becomes tangent to the curve $y = \frac{c}{x+1}$.

Sol. Equation of line joining A & B is $x + y = 4$

Solving this line and curve

we get $\qquad 4 - x = \frac{c}{x+1}$

$$x^2 - 3x + (c - 4) = 0 \qquad \text{......(i)}$$

For tangency, roots of this equation must be equal.

Hence, discriminant of quadratic equation = 0

$$9 = 4(c - 4)$$

$$\frac{9}{4} = c - 4$$

$$c = \frac{9}{4} + 4$$

$$c = \frac{25}{4}$$

$$x^2 - 3x + \frac{9}{4} = 0$$

$$x^2 - 2.\,\frac{3}{2}x + \frac{9}{4} - \frac{9}{4} + \frac{9}{4} = 0$$

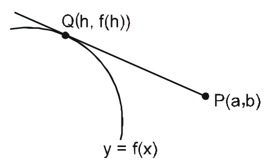
$$\left(x-\frac{3}{2}\right)^2 = 0$$

$$x = \frac{3}{2}$$

Hence, point of contact becomes $\left(\frac{3}{2}, \frac{5}{2}\right)$

Tangent and Normal from an external point:

For a point P (a, b) not situated on the curve y = f (x) the equation of potential tangents to the curve y = f (x), which pass through (a, b) can be obtained by solving for the point of contact Q.

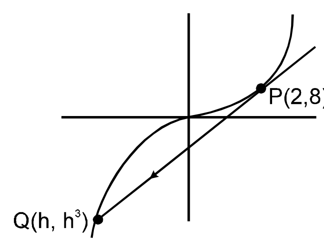


Figure

$$f'(h) = \frac{f(h)-b}{h-a}$$

And equation of tangent is $y - b = \frac{f(h)-b}{h-a}(x - a)$

Ex.6 Determine the coordinates of point Q, where the tangent at P (2, 8) on the curve $y = x^3$ intersects the curve once more.



Sol. Equation of tangent at (2, 8) is $y = 12x - 16$

Solving this with $y = x^3$

$$x^3 - 12x + 16 = 0$$

This cubic will give all points of intersection of line and Curve $y = x^3$ i.e., point P and Q. (see figure)

But, since line is tangent at P

So $x = 2$ will be a repeated root of equation $x^3 - 12x + 16 = 0$

And another root will be $x = h$.

Using theory of equations:

Sum of roots

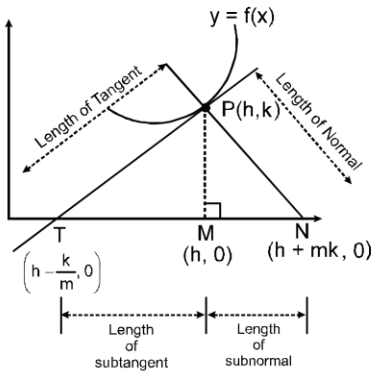
$$2 + 2 + h = 0$$

$$h = -4$$

Hence, coordinates of Q are (– 4, – 64)

Lengths of tangent, normal, sub tangent and subnormal:

Consider any point P (h, k) on the curve y = f (x). Draw a tangent at point P which intersects the x – axis at T, and draw a normal at point P, which intersects the x – axis at N. the distance PT is referred to as the length of the tangent and the distance PN is revered to as the length of the tangent, and the distance PN is referred to as the length of the normal (as illustrated in the figure.)



Projection of segment PT on x-axis, TM, is called the sub tangent and similarly projection of line segment PN on x axis, MN is called subnormal.

Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is m (x – h) = (y – k).

Putting y = 0, we get x - intercept of tangent is $x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is x = h + km

Now, length PT, PN,TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of sub tangent

(iv) MN = |km| = Length of subnormal

Ex.7 Determine the length of the tangent to the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.

Sol.
$$m = \left.\frac{dy}{dx}\right|_{x=0}$$

$$\frac{dy}{dx} = 3x^2 + 6x + 4$$

$$m = 4$$

$$k = y\,(0)$$

$$k = -1$$

$$\text{In} = |k|\sqrt{1 + \frac{1}{m^2}}$$

$$\text{In} = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$$

Ex.8 Find the value of 'p' such that the lengths of the sub-tangent and sub-normal are equal for the curve $y = e^{px} + px$ at the point (0, 1).

Sol. $\frac{dy}{dx} = pe^{px} + p$ at point $(0, 1) = 2p$

$$\text{Subnormal} = \left|y\frac{dy}{dx}\right|$$

$$\text{Sub tangent} = \left|y\frac{dx}{dy}\right|$$

$$\frac{dy}{dx} = \pm 1$$

$$2p = \pm 1$$

$$p = \pm\frac{1}{2}$$

Ex.9 Demonstrate that, for the curve $y = a \text{ In } (x^2 - a^2)$ the sum of lengths of the tangent and sub tangent at any point is proportional to the coordinates of the point of tangency.

Sol. Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{2ax_1}{x^2_1 - a^2}$$

Length of tangent + sub tangent

$$= |y_1|\sqrt{1+\frac{1}{m^2}} + \left|\frac{y_1}{m}\right|$$

$$= |y_1| \frac{\sqrt{x_1^{\ 4} + a^4 + 2a^2x_1^{\ 2}}}{2|ax_1|} + \left|\frac{y_1(x_1^{\ 2} - a^2)}{2ax_1}\right|$$

$$= \left|\frac{y_1(x_1^{\ 2} + a^2)}{2ax_1}\right| + \left|\frac{y_1(x_1^{\ 2} - a^2)}{2ax_1}\right|$$

$$= \frac{|y_1|(2x_1^{\ 2})}{2|ax_1|} = \left|\frac{x_1 \ y_1}{a}\right|$$