APPLICATIONS OF DERIVATIVES

INTRODUCTION, RATE OF CHANGE OF QUANTITIES

Derivative as rate of change:

In different applied mathematical domains, there is a pursuit to understand the rate at which one variable changes concerning another. While the rate of change commonly relates to time, it can also involve other variables. For instance, an economist might examine how investment fluctuates in relation to variations in interest rates, and a physician may seek to understand how minor changes in dosage impact the body's response to a drug.

Rate of Change of Quantities

If $(y = f(x))$ represents a function of x, and x is a function of time, then y is also a function of time, denoted as t.

Now $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\text{rat of change of y}}{\text{rat of change of x}}$$

$\frac{dy}{dx}$ Equals the quotient of the rate of change of y and the rate of change of x.

$\frac{dy}{dx}$ Symbolizes the instantaneous rate of change in y concerning x.

VELOCITY AND ACCELERATION

$$\text{Velocity} = \frac{ds}{dt} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} \text{ and acceleration} = \frac{d^2 s}{dt^2}$$

Where s is the distance covered in time t and $s + \delta s$ is the distance covered in time $t + \delta t$ this represents the approximate change in the value of the function

$$y = f(x) \quad \delta y = \frac{dy}{dx} \delta x$$

Where δy = approximate change in y.

Approximate value of function

$$y = f(x) \text{ at } x = a,$$

$$f(a + h) = f(a) + hf'(a)$$

Where $h \to 0$

Ex.1 Determine the rate at which the area of a circle increases when its radius is 5 cm in relation to:

(i) With respect to radius

(ii) With respect to diameter

Sol. (i) $A = \pi r^2, \quad \frac{dA}{dr} = 2\pi r$

$\therefore$ $\left.\frac{dA}{dD}\right]_{r=5} = 10\pi \text{ cm}^2/\text{cm}$

(ii) $A = \frac{\pi}{4} D^2, \quad \frac{dA}{dD} = \frac{\pi}{2} D$

$\therefore$ $\left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} . 10 = 5\pi \text{ cm}^2/\text{cm}.$

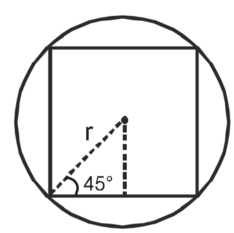
Ex.2 If the area of a circle is increasing at a rate of 2 cm^2/sec, determine the rate at which the area of the inscribed square is increasing.

Sol. Area of circle, $A_1 = \pi r^2$.

Area of square, $A_2 = 2r^2$ (see figure)

$$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}$$

$\Rightarrow$ $\frac{dA_2}{dt} = 4r.\frac{dr}{dt}$



$\because$ $2 = 2\pi r . \frac{dr}{dt}$

$\Rightarrow$ $r\frac{dr}{dt} = \frac{1}{\pi}$

$\therefore$ $\frac{dA_2}{dt} = 4 . \frac{1}{\pi} = \frac{4}{\pi} \text{ cm}^2/\text{sec}$

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm^2/sec.

Ex.3 The volume of a cube is increasing at a rate of 7 cm^3/sec. Determine the rate at which the surface area is increasing when the length of an edge is 4 cm.

Sol. Let at some time t, the length of edge is x cm.

$$v = x^3$$

$\Rightarrow$ $$\frac{dv}{dt} = 3x^2\frac{dx}{dt} \qquad \left(\text{but } \frac{dv}{dt} = 7\right)$$

$\Rightarrow$ $$\frac{dx}{dt} = \frac{7}{3x^2} \text{ cm/sec.}$$

Now $$S = 6x^2$$

$$\frac{dS}{dt} = 12x\frac{dx}{dt}$$

$\Rightarrow$ $$\frac{dS}{dt} = 12x.\ \frac{7}{3x^2} = \frac{28}{x}$$

when $$x = 4 \text{ cm},$$

$$\frac{dS}{dt} = 7 \text{ cm}^2\text{/sec.}$$

Ex.4 Sand is flowing from a pipe at a rate of 12 cm^3/s. As the falling sand shapes a cone on the ground, the height of the cone is consistently one-sixth of the radius of its base. Determine the rate at which the height of the sand cone is increasing when the height is 4 cm.

Sol. $$V = \frac{1}{3}\pi r^2 h$$

but $$h = \frac{r}{6}$$

$\Rightarrow$ $$V = \frac{1}{3}\ \pi\,(6h)^2\ h$$

$\Rightarrow$ $$V = 12\pi\ h^3$$

$\Rightarrow$ $$\frac{dv}{dt} = 36\pi\ h^2.\ \frac{dh}{dt}$$

when, $\frac{dv}{dt} = 12\ cm^3/s$

and $h = 4$ cm

$$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}\ \text{cm/sec.}$$