APPLICATIONS OF DERIVATIVES

MAXIMA & MINIMA

Global Maximum:

A function f(x) is deemed to possess a global maximum over a set E if there is at least one c in E such that **$f(x) \leq f(c)$ for all $x \in E$.**

The occurrence of a global maximum is identified at $x = c$, and the corresponding value is denoted as f(c).

Local Maxima:

A function f(x) is characterized as having a local maximum at $x = c$ if the value f(c) represents the highest point of the function within a small interval $(c - h, c + h)$, where $h > 0$, centered around c.

i.e. For all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \leq f(c)$.

Note:

If $x = c$ is situated at a boundary point, then appropriately consider either the interval $(c - h, c)$ or $(c, c + h)$ with $h > 0$.

Global Minimum:

A function f(x) is considered to possess a global minimum over a set E if there is at least one c in E such that **$f(x) \geq f(c)$ for all $x \in E$.**

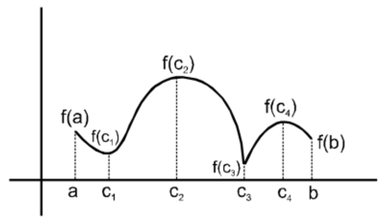
Local Minima:

A function f(x) is characterized as having a local minimum at $x = c$ if the value f(c) represents the smallest point of the function within a small interval $(c - h, c + h)$, where $h > 0$, centered around c.

i.e. For all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

Extrema:

An extremum refers to either a maximum or a minimum value.



Explanation:

Consider graph of $y = f(x)$, $x \in [a, b]$

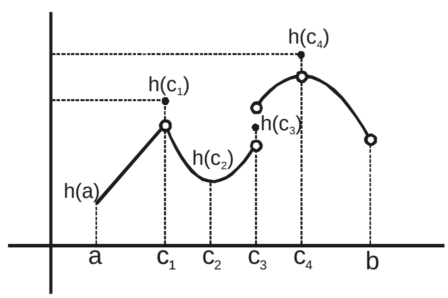
At $x = a$, $x = c_2$, and $x = c_4$, the function has **local maxima** with corresponding maximum values $f(a)$, $f(c_2)$, and $f(c_4)$, respectively.

At $x = c_1$, $x = c_3$, and $x = b$, the function exhibits **local minima** with respective minimum values $f(c_1)$, $f(c_3)$, and $f(b)$.

The point $x = c_2$ represents a **global maximum**.

The point $x = c_3$ signifies a **global minimum.**

Consider the graph of $y = h(x)$, $x \in [a, b)$



At $x = c_1$ and $x = c_4$, the function has local maxima with corresponding maximum values $h(c_1)$ and $h(c_4)$, respectively.

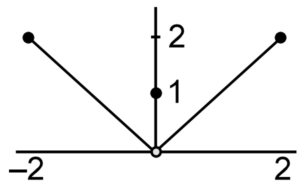
At $x = a$ and $x = c_2$, the function exhibits local minima with minimum values $h(a)$ and $h(c_2)$, respectively. However, $x = c_3$ does not represent a point of either maximum or minimum.

The global maximum corresponds to the value $h(c_4)$.

The global minimum is represented by the value $h(a)$.

Ex.1 Consider the function $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behavior of f(x) at x = 0.

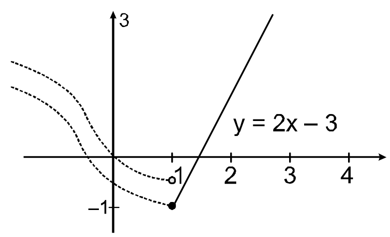
Sol. The function f(x) exhibits local maxima at x = 0, as depicted in the figure.



Ex.2 Let
$$f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$$

Determine all potential values of b for which f(x) attains its minimum value at x = 1.

Sol. These types of problems can be readily resolved using a graphical approach, as illustrated in the figure.



Therefore, the limit of f(x) from the left of x = 1 should either be greater than or equal to the value of the function at x = 1.

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

$$-1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$$

$$\frac{(b^2 + 1)(b - 1)}{(b + 1)\ (b + 2)} \ge 0$$

$$b \in (-2, -1) \cup [1, +\infty)$$

Maxima, Minima for differentiable functions:

The mere definition of maxima and minima can be cumbersome in problem-solving. To overcome this difficulty, we employ derivatives as a tool.

(i) A necessary condition for an extrema:

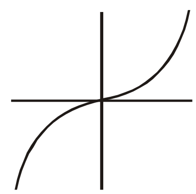
Consider the function f(x) to be differentiable at x = c.

Theorem:

A prerequisite for f(c) to be an extremum of f(x) is that the derivative, f'(c), equals zero.

f(c) is extremum

$f'(c) = 0$



Note: $f'(c) = 0$ is only a necessary condition but not sufficient

$f'(c) = 0$

f(c) is extremum.

Consider $f(x) = x^3$

$f'(0) = 0$

But f(0) is not an extremum (see figure).

(ii) Sufficient condition for an extrema:

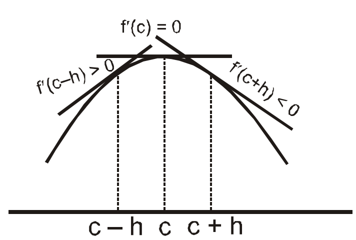
Consider f(x) as a function that is differentiable.

Theorem:

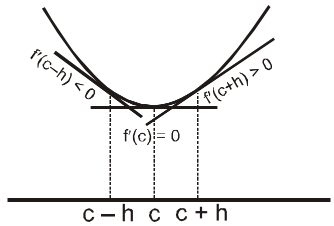
A condition that is adequate for f(c) to be an extremum of f(x) is the change in sign of the derivative, f'(x), as x passes through c.

The value f(c) represents an extremum, as illustrated in the figure.

The sign of the derivative, f'(x), changes as x passes through c.



The point x = c corresponds to a maximum, as the derivative f'(x) changes sign from positive to negative.



The point x = c signifies a local minimum, as shown in the figure, where the derivative f'(x) changes sign from negative to positive.

Stationary points:

The locations on the graph of the function f(x) where the derivative f'(x) equals zero are termed stationary points. At a stationary point, the rate of change of f(x) is zero.

Ex.3 Determine the stationary points of the function.$f(x) = 4x^3 - 6x^2 - 24x + 9$.

Sol.

$$f'(x) = 12x^2 - 12x - 24$$

$$f'(x) = 0$$

$$x = -1, 2$$

$$f(-1) = 23, f(2) = -31$$

$$(-1, 23), (2, -31) \text{ are stationary points}$$

Ex.4 If $f(x) = x^3 + ax^2 + bx + c$ exhibits extreme values at $x = -1$ and $x = 3$, ascertain the values of a, b, and c.

Sol. Extreme values essentially refer to maximum or minimum values. Given that f(x) is a differentiable function,

$$f'(-1) = 0 = f'(3)$$

$$f'(x) = 3x^2 + 2ax + b$$

$$f'(3) = 27 + 6a + b = 0$$

$$f'(-1) = 3 - 2a + b = 0$$

$$a = -3, b = -9, c \in R$$

First Derivative Test:

Extreme values essentially refer to maximum or minimum values. Given that f(x) is a differentiable function,

Step I.

Determine the derivative of f(x) with respect to x.

Step II.

Solve the equation $f'(x) = 0$, and let $x = c$ be one of the solutions. This corresponds to finding stationary points.

Step III. Examine the change in sign.

(i) If the sign of $f'(x)$ shifts from negative to positive as x traverses c from left to right, then $x = c$ represents a point of local minimum.

(ii) If the sign of $f'(x)$ shifts from positive to negative as x traverses c from left to right, then $x = c$ indicates a point of local maximum.

(iii) If the sign of $f'(x)$ remains unchanged as x crosses c, then $x = c$ is neither a point of maxima nor minima.

Ex.5 Determine the points of maxima or minima for the function $f(x) = x^2(x - 2)^2$.

Sol.

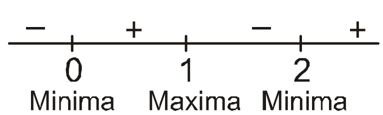
$$f(x) = x^2 (x - 2)^2$$

$$f'(x) = 4x (x - 1) (x - 2)$$

$$f'(x) = 0$$

$$x = 0, 1, 2$$

Observing the change in sign of $f'(x)$.



Therefore, $x = 1$ is a point of maximum, while $x=0$ and $x=2$ are points of minimum.

Note: For continuous functions, points of maxima and minima occur alternately.

Ex.6 Determine the points of maxima and minima for the function $f(x) = x^3 - 12x$. Additionally, sketch the graph of this function.

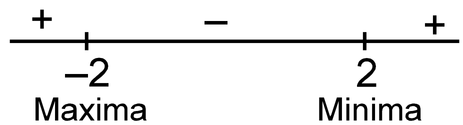
Sol.

$$f(x) = x^3 - 12x$$

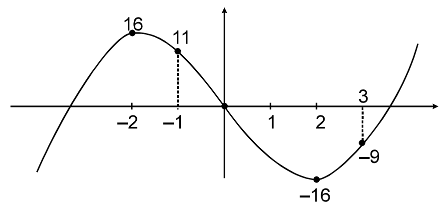
$$f'(x) = 3(x^2 - 4) = 3(x - 2)\ (x + 2)$$

$$f'(x) = 0$$

$$x = \pm 2$$



To plot the graph, let's identify the maximum and minimum values of f(x).



x	f(x)
2	–16
–2	+16

Ex.7 Demonstrate that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not possess any points of local maxima or minima. Subsequently, sketch the graph.

Sol.

$$f(x) = x^3 - 6x^2 + 12x - 8$$

$$f'(x) = 3(x^2 - 4x + 4)$$

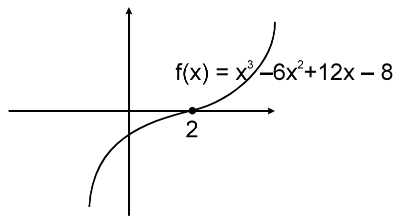
$$f'(x) = 3(x - 2)^2$$

$$f'(x) = 0$$

$$x = 2$$

But, clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$.

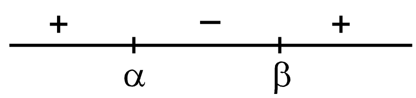
Therefore, f(x) lacks any points of maxima or minima. Indeed, f(x) is a monotonically increasing function for all x in the real numbers.



Ex.8 Consider the function $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) determine the potential values of 'a' if the function has a positive point of maxima.

Sol.
$$f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$$

Let α,and β be roots of $f'(x) = 0$ with α being the smaller root. Observe the changes in sign of $f'(x)$.



The maximum occurs at the smaller root α which must be positive. This implies that both roots of $f'(x) = 0$must be positive and distinct.

(i) $D > 0 \quad \Rightarrow \quad a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0 \quad \Rightarrow \quad a < 7$

(iii) $f'(0) > 0 \quad \Rightarrow \quad a \in (-\infty, -3) \cup (3, \infty)$

From (i), (ii) and (iii)

$$\Rightarrow \quad a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$$

Application of Maxima, Minima:

In a given problem, an objective function can be formulated in terms of a single parameter, and the extremum value can be determined by setting the derivative equal to zero. As explained in the nth derivative test, maxima/minima can be recognized.

(i) Useful Formulae of Mensuration to Remember: (For competitive exam)

(a) Volume of a cuboid = lbh.

(b) Surface area of cuboid = 2(lb + bh + hl).

(c) Volume of cube = a^3

(d) Surface area of cube = $6a^2$

(e) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(f) Curved surface area of cone = πrl (l = slant height)

(g) Curved surface area of a cylinder = $2\pi rh$.

(h) Total surface area of a cylinder $= 2\pi rh + 2\pi r^2$.

(i) Volume of a sphere $= \pi r^3$.

(j) Surface area of a sphere $= \frac{4}{3}\ 4\pi r^2$.

(k) Area of a circular sector $= \frac{1}{2}\ r^2\ \theta$, when θ is in radians.

(l) Volume of a prism = (area of the base) × (height).

(m) Lateral surface area of a prism = (perimeter of the base) × (height).

(n) The total surface area of a prism is equal to the sum of the lateral surface area and twice the area of the base.

(Keep in mind that the lateral surfaces of a prism are all rectangles.)

(o) Volume of a pyramid $= \frac{1}{3}$ (area of the base) × (height).

(p) Curved surface area of a pyramid $= \frac{1}{2}$ (perimeter of the base) × (slant height).

(Keep in mind that the inclined surfaces of a pyramid are triangular in shape.)

Ex.9 If the equation $x^3 + px + q = 0$ possesses three real roots, demonstrate that $4p^3 + 27q^2 < 0$..

Sol.

$$f(x) = x^3 + px + q,$$

$$f'(x) = 3x^2 + p$$

f (x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$$x = \pm\sqrt{\frac{-p}{3}},\quad p < 0$$

f is maximum at $x = -\sqrt{\frac{-p}{3}}$ and minimum at $x = \sqrt{\frac{-p}{3}}$

$$f\left(-\sqrt{\frac{-p}{3}}\right)\ f\ \left(\sqrt{\frac{-p}{3}}\right) < 0$$

$$\left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\ \left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$$

$$q^2 + \frac{4p^3}{27} < 0,\ 4p^3 + 27q^2 < 0.$$

Ex.10 Determine two positive numbers, x and y, such that $x + y = 60$, and the expression xy^3 is maximized.

Sol.
$$x + y = 60$$
$$x = 60 - y$$
$$xy^3 = (60 - y)\, y^3$$
$$f(y) = (60 - y)\, y^3\ ;\ y \in (0, 60)$$

For maximizing f(y) let us find critical points

$$f'(y) = 3y^2 (60 - y) - y^3 = 0$$
$$f'(y) = y^2 (180 - 4y) = 0$$
$$y = 45$$
$$f'(45^+) < 0 \text{ and } f'(45^-) > 0.$$

Hence local maxima at $y = 45$.

$$x = 15 \text{ and } y = 45.$$

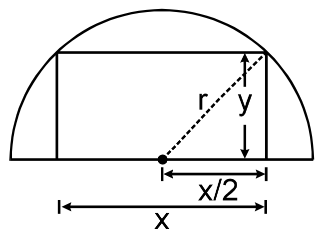
Ex.11 Determine the rectangle with the maximum area that can be inscribed inside a semicircle of radius 'r'.

Sol. Consider the dimensions of the rectangle as x and y, as depicted in the figure.

$$A = xy.$$

Here, x and y are not independent variables; they are related by the Pythagorean theorem with the radius 'r'.

$$\frac{x^2}{4} + y^2 = r^2$$
$$y = \sqrt{r^2 - \frac{x^2}{4}}$$
$$A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$$
$$A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$$
$$f(x) = r^2x^2 - \frac{x^4}{4}\ ; \qquad x \in (0, r)$$



A(x) is maximum when f(x) is maximum

$$f'(x) = x(2r^2 - x^2) = 0$$

$$x = r\sqrt{2}$$

$$f'(r\sqrt{2^+}) < 0 \quad \text{and} \quad f'(r\sqrt{2^-}) > 0$$

Confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

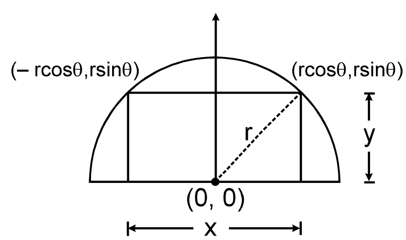
Aliter:

Let's select a coordinate system with the origin at the center of the circle, as depicted in the figure.

$$A = xy$$

$$A = 2\,(r\cos\theta)\,(r\sin\theta)$$

$$A = r^2 \sin 2\theta, \quad \theta \in \left(0, \frac{\pi}{2}\right)$$



Clearly, A is maximum when $\theta = \frac{\pi}{4}$

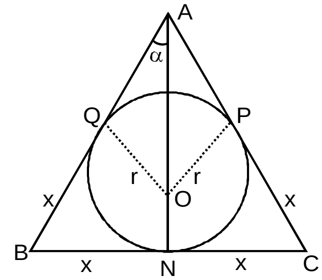
$$x = r\sqrt{2} \text{ and } y = \frac{r}{\sqrt{2}}.$$

Ex.12 Demonstrate that the minimum perimeter of an isosceles triangle circumscribed about a circle of radius 'r' is. $6\sqrt{3}\ r$.

Sol.

$$AQ = r\cot\alpha = AP$$

$$AO = r\,\text{cosec}\alpha$$



$$\frac{x}{AO+ON} = \tan\alpha$$

$$x = (r\ \text{cosec}\alpha + r)\ \tan\alpha$$

$$x = r(\sec\alpha + \tan\alpha)$$

$$\text{Perimeter} = p = 4x + 2AQ$$

$$p = 4r(\sec\alpha + \tan\alpha) + 2r\cot\alpha$$

$$p = r(4\sec\alpha + 4\tan\alpha + 2\cot\alpha)$$

$$\frac{dp}{d\alpha} = r[4\sec\alpha\ \tan\alpha + 4\sec^2\alpha - 2\text{cosec}^2\alpha]$$

For max or min $\frac{dp}{d\alpha} = 0$

$$2\sin^3\alpha + 3\sin^2\alpha - 1 = 0$$

$$(\sin\alpha + 1)\ (2\sin^2\alpha + \sin\alpha - 1) = 0$$

$$(\sin\alpha + 1)^2\ (2\sin\alpha - 1) = 0$$

$$\sin\alpha = \frac{1}{2}$$

$$\alpha = 30° = \frac{\pi}{6}$$

$$\text{Pleats} = r\left[\frac{4.2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + 2\sqrt{3}\right]$$

$$r\left[\frac{8+4+6}{\sqrt{3}}\right]$$

$$r\ \frac{\left(6\sqrt{3}\sqrt{3}\right)}{\sqrt{3}} = 6\sqrt{3}\ r$$

Ex.13 Determine the dimensions of a right circular cylinder inscribed in a given cone such that its volume is maximized.

Sol. Let x be the radius of cylinder and y be its height $v = \pi x^2 y$

x, y can be related by using similar triangles (as shown in figure).

$$\frac{y}{r-x} = \frac{h}{r}$$

$$y = \frac{h}{r}\ (r - x)$$

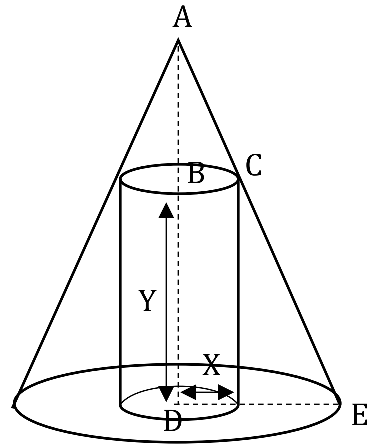
$$v(x) = \pi x^2 \frac{h}{r}(r - x) \quad x \in (0, r)$$

$$v(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$v'(x) = \frac{\pi h}{r} x(2r - 3x)$$

$$v' = \left(\frac{2r}{3}\right) = 0$$

and $$v''\left(\frac{2r}{3}\right) < 0$$



Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Note: The formulas for the volume and surface area of significant solids are highly valuable in problems involving maxima and minima.

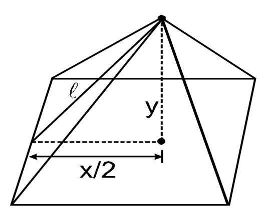
Ex.14 Among all regular square pyramids with a volume of $36\sqrt{2}$ cm^3, determine the dimensions of the pyramid that has the smallest lateral surface area.

Sol. Consider the length of one side of the base as x cm, and let y represent the perpendicular height of the pyramid (refer to the figure).

$$V = \frac{1}{3} \times \text{ area of base x height}$$

$$V = \frac{1}{3} x^2 y = 36\sqrt{2}$$

$$y = \frac{108\sqrt{2}}{x^2}$$



$$S = \frac{1}{2} \times \text{perimeter of base x slant height} = \frac{1}{2}(4x).\lambda$$

$$\lambda = \sqrt{\frac{x^2}{4} + y^2}$$

$$S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$$

$$S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$$

$$S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$$

$$f(x) = x^4 + \frac{8.(108)^2}{x^2} \quad \text{for minimizing } f(x)$$

$$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0$$

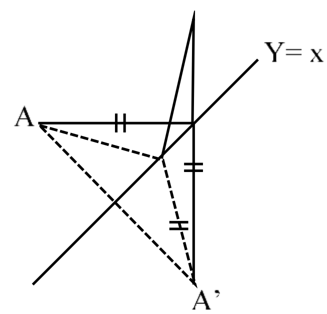
$$f'(x) = 4\ \frac{(x^6 - 6^6)}{x^3} = 0$$

$x = 6$, which a point of minima. Hence $x = 6$ cm and $y = 3\sqrt{2}$.

Ex.15 Choose two fixed points A(1, 2) and B(–2, –4). Select a variable point P on the line $y = x$ in such a way that the perimeter of triangle PAB is minimized. Determine the coordinates of P.

Sol. As the distance AB remains constant, the key to minimizing the perimeter of triangle PAB is essentially to minimize the sum (PA + PB).

Let A' be the mirror image of A across the line $y = x$ (refer to the figure).



$$F(P) = PA + PB$$

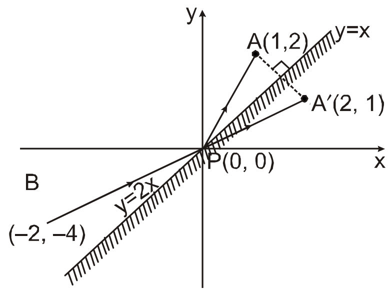
$$F(P) = PA' + PB$$

But for $\Delta PA'B$

$PA' + PB \geq A'B$ and equality is achieved when P, A', and B are collinear.

Hence, for the minimum path length, point P is the specific location at which PA and PB transform into incident and reflected rays concerning the mirror $y = x$.

The equation of the line connecting A' and B is $y = 2x$. The intersection of this line with $y = x$ gives the point P. Therefore, P is equivalent to (0, 0).



Note: The aforementioned concept is highly valuable, as problems of this nature would become cumbersome if one were to formulate the perimeter as a function of the position of point P and then attempt to minimize it.