APPLICATIONS OF DERIVATIVES

INCREASING & DECREASING FUNCTIONS

Monotonicity of a function:

Consider a real-valued function (f) with a domain (D (DR)), and let (S) be a subset of (D). The function (f) is defined to be monotonically increasing (non-decreasing) (increasing) in (S) if for every.

$$x_1, x_2 \in S, x_1 < x_2$$

$$f(x_1) \leq f(x_2).$$

The function f is considered monotonically decreasing (non-increasing) (decreasing) in S if, for every...

$$x_1, x_2 \in S, x_1 < x_2$$

$$f(x_1) \geq f(x_2)$$

The function f is termed strictly increasing in S if, for every...

$$x_1, x_2 \in S, x_1 < x_2$$

$$f(x_1) < f(x_2).$$

Likewise, f is described as strictly decreasing in S if, for every...

$$x_1, x_2 \in S, x_1 < x_2$$

$$f(x_1) > f(x_2).$$

Notes:-

(i) If f is strictly increasing, it implies that f is monotonically increasing (non-decreasing). However, the converse need not be true.

(ii) If f is strictly decreasing, it implies that f is monotonically decreasing (non-increasing). Again, the converse need not be true.

(iii) If f(x) is a constant in S, then f is both increasing and decreasing in S.

(iv) A function f is termed an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing.

(v) f is considered a monotonic function if it is either monotonically increasing or monotonically decreasing.

(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non-monotonic in S.

Application of differentiation for detecting monotonicity:

Let I be an interval (open, closed, or semi-open and semi-closed):

(i) If $f'(x) > 0$ for all x in I, then f is strictly increasing in I.

(ii) If $f'(x) < 0$ for all x in I, then f is strictly decreasing in I.

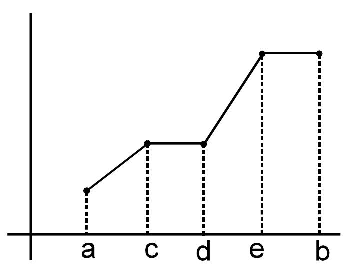
Notes:-

Consider I as an interval or ray, which is a subset of the domain of f. If...

If $f'(x) > 0$ for all x in I, except for countably many point where $f'(x) = 0$, them $f'(x)$ is strictly increasing in I.

If $f'(x) = 0$ at countably many points, it implies that $f'(x) = 0$ does not occur on an interval that is a subset of I.

Consider another function whose graph is depicted below for x in the interval (a, b).



In this case as well, $f'(x) \geq 0$ for all x in (a, b). However it's important to observe that in this scenario $f'(x) = 0$ is true for all x in (c, d) and (e, b).

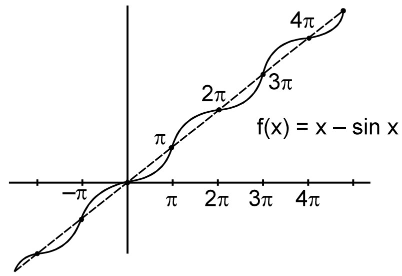
Ex.1 Consider the function $f'(x) = x - \sin(x)$. Determine the intervals of monotonicity.

Sol. $$f'(x) = 1 - \cos x$$

Now, $f'(x) > 0$ everywhere, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc.

However, all these points are distinct (countable) and do not constitute an interval.

Therefore, we can infer that f(x) is strictly increasing across the entire real number line. This can be visually observed in the graph as well.



Ex.2 Determine the minimum value of K for which the function $x^2 + kx + 1$ is an increasing function in the interval $1 < x < 2$.

Sol. $$f(x) = x^2 + kx + 1$$

For f(x) to be increasing, $f'(x) > 0$

$$\Rightarrow \frac{d}{dx}(x^2 + kx + 1) > 0$$

$$2x + k > 0$$

$$\Rightarrow k > -2x$$

For $x \in (1, 2)$ the least value of k is –2

Ex.3 Determine the intervals in which $f(x) = x^3 - 3x + 2$ is increasing.

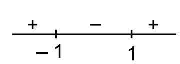
Sol. $$f(x) = x^3 - 3x + 2$$

$$\Rightarrow f'(x) = 3(x^2 - 1)$$

$$\Rightarrow f'(x) = 3(x - 1)(x + 1)$$

for M.I. $f'(x) \geq 0$

$$\Rightarrow 3(x - 1)(x + 1) \geq 0$$



$\Rightarrow$ $x \in (-\infty, -1] \cup [1, \infty)$, thus f is increasing in $(-\infty, -1]$ and also in $[1, \infty)$

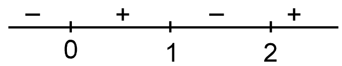
Ex.4 Determine the intervals of monotonic behavior for the given functions.

(i) $f(x) = x^2 (x - 2)^2$ (ii) $f(x) = x \ln x$ (iii) $f(x) = \sin x + \cos x$; $x \in [0, 2\pi]$

Sol. (i) $$f(x) = x^2 (x - 2)^2$$

$$f'(x) = 4x (x - 1)(x - 2)$$

Observing the sign change of $f'(x)$



Hence increasing in [0, 1] and in [2, ∞) and decreasing for

$x \in (-\infty, 0]$ and [1, 2]

(ii) $f(x) = x \ln x$

$\Rightarrow$ $f'(x) = 1 + \lambda n\, x$

$f'(x) \geq 0$

$\Rightarrow$ $\ln x \geq -1$

$\Rightarrow$ $x \geq \frac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\frac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \frac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

For increasing $f'(x) \geq 0$

$\Rightarrow$ $\cos x \geq \sin x$

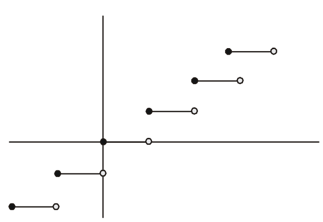
$\Rightarrow$ f is increasing in $\left[0. \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$ f is decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Notes:-

If a function f (x) demonstrate increasing behavior in the open interval (a, b) and is continues over the closed interval [a, b] then f (x) is increasing across the entire interval [a, b]

Ex.5 The function f (x) = [x] is a step function is it strictly increasing for all x in the set of real numbers?

Sol. No, f (x) = [x] is an increasing (monotonically increasing or non – decreasing) function, but it is not stricly increasing, as evidenced by its graph.



Ex.6 If f (x) = $\sin^4 x + \cos^4 x + bx + c$, determine the potential value of b and c such that f (x) is monotonic for all x in the set of real number.

Sol. $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4\sin^3x \cos x - 4\cos^3x \sin x + b$

$= -\sin 4x + b.$

Case (i) : for M.I. $f'(x) \geq 0$ for all $x \in R$

$b \geq \sin 4x$ for all $x \in R$

$\Leftrightarrow$ $b \geq 1$

Case (ii) : for M.D. $f'(x) \leq 0$ for all $x \in R$

$\Leftrightarrow$ $b \leq \sin 4x$ for all $x \in R$

$\Leftrightarrow$ $b \leq -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Ex.7 Determine the potential value of 'a' for which f (x) = $e^{2x} - (a + 1) e^x + 2x$ is monotonically increasing for all x in the set of real number.

Sol. $f(x) = e^{2x} - (a + 1) e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1) e^x + 2$

Now, $2e^{2x} - (a + 1) e^x + 2 \geq 0$ for all $x \in R$

$\Rightarrow$ $2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \geq 0$ for all $x \in R$

$(a + 1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow$ $a + 1 \leq 4$ ($e^x + \frac{1}{e^x}$ has minimum value 2)

$\Rightarrow$ $a \leq 3$

Aliter (Using graph)

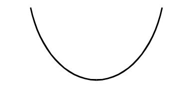
$2e^{2x} - (a + 1) e^x + 2 \geq 0$ for all $x \in R$

Putting $e^x = t$; $t \in (0, \infty)$

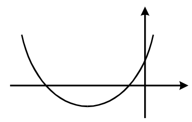
$2t^2 - (a + 1) t + 2 \geq 0$ for all $t \in (0, \infty)$

Case - (i) $D \le 0$

$\Rightarrow$ $(a+1)^2 - 4 \le 0$



$\Rightarrow$ $(a+5)(a-3) \le 0$



$\Rightarrow$ $a \in [-5, 3]$

Case - (ii) : both roots are non-positive

$D \ge 0$ & $-\frac{b}{2a} < 0$ & $f(0) \ge 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $\frac{a+1}{4} < 0$ & $2 \ge 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $a < -1$ & $a \in R$

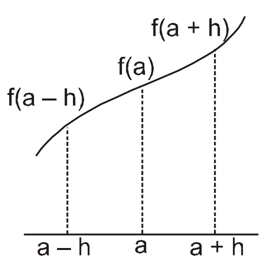
$\Rightarrow$ $a \in (-\infty, -5]$

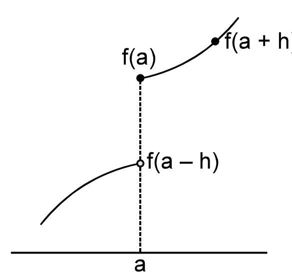
Taking union of (i) and (ii),

We get, $a \in (-\infty, 3]$

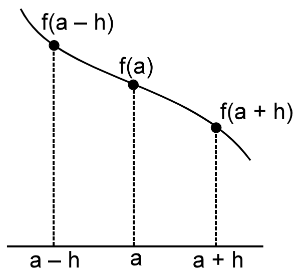
Monotonicity of function about a point:-

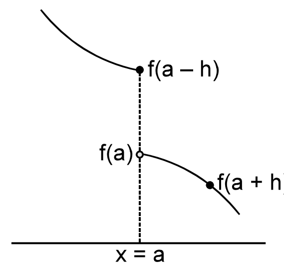
(i) A function f (x) is referred to as a strictly increasing function around a point (or at a point) $a \in D_f$ if it is strictly increasing in a open interval containing a (as depicted in the figure.)





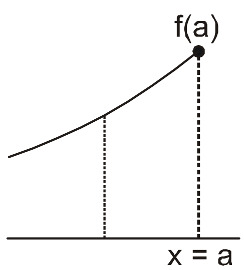
(ii) A function f(x) is termed a strictly decreasing function around the point x = a if it exhibits strict decrease within an open interval that includes a (as illustrated in the figure).



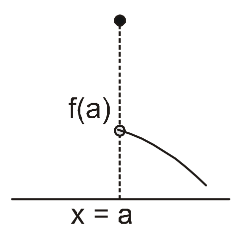


Notes:-

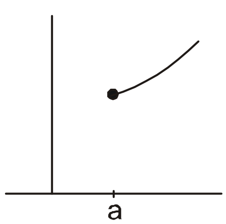
If x = a is a boundary point, employ the appropriate one-sided inequality to examine the monotonicity of f(x).



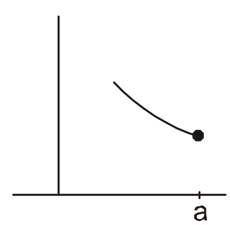
f is increasing at x = a



f is decreasing at x = a



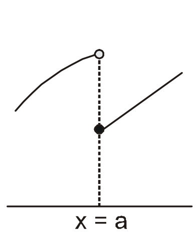
f is increasing at x = a



f is decreasing at x = a

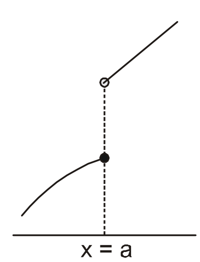
E.g. At the given value x = a, determine whether the depicted functions (as illustrated in the figure) are increasing, decreasing, or exhibiting neither an increase nor decrease.

(i)

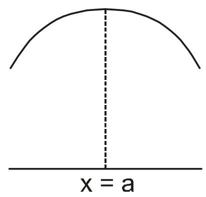


neither increasing nor decreasing at x = a

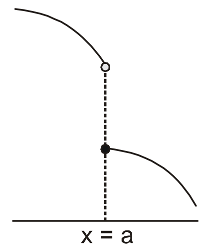
(ii)



Increasing at x = a

(iii) 

neither increasing nor decreasing at x = a

(iv) 

decreasing at x = a

Test for increasing and decreasing functions about a point :

Consider the differentiability of the function f (x):

(i) If $f'(a) > 0$, then f (x) is increasing at x = a.

(ii) If $f'(a) < 0$, then f (x) is decreasing at x = a.

(iii) If $f'(a) = 0$, analyze the sign of $f'(x)$ in the left and right neighborhoods of a:

(a) If $f'(x)$ is positive in both neighborhoods then f is increasing at x = a.

(b) If $f'(x)$ is negative in both neighborhoods, then f is decreasing at x = a.

(c) If $f'(x)$ has opposite signs in these neighborhoods, then f is non – monotonic at x = a.

Ex.8 Consider the function $f(x) = x^3 - 3x + 2$ investigate the monotonic behavior of the function at the points x = 0, 1, 2.

Sol.

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3(x^2 - 1)$$

(i) $f'(0) = -3$

$\Rightarrow$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighborhood and $f'(x)$ is negative in right neighborhood.

$\Rightarrow$ neither increasing nor decreasing at x = 1.

(iii) $f'(2) = 9$

$\Rightarrow$ increasing at x = 2

Notes:-

This method is applicable exclusively to functions that exhibit continuity at x = a.