# PROBABILITY

## RANDOM VARIABLE & IT PROBABILITY DISTRIBUTION

## Random Variables

A variable is a quantity that can change its value, varying with different outcomes of an experiment. If a variable's value depends on the outcome of a random experiment, it is termed a random variable and can assume any real value.

In mathematical terms, a random variable is a real-valued function defined on the sample space S of a random experiment. Denoted by capital letters such as X, Y, M, etc., it is represented by lowercase letters like x, y, z, m, etc., for specific values.

Consider the random experiment of tossing a coin 20 times, where earning Rs. 5 is associated with getting heads and losing Rs. 5 with tails. In a competition to see who can earn more money between you and your friend, the value of getting heads in the coin toss for 20 times can range from zero to twenty. If we let X denote the number of heads, then $X = \{0, 1, 2, \ldots, 20\}$. The probability of getting heads is consistently $\frac{1}{2}$.

## Properties of a Random Variable

- It exclusively accepts real values.
- If X is a random variable and C is a constant, then CX is likewise a random variable.
- When $X_1$ and $X_2$ are two random variables, both $X_1 + X_2$ and $X_1 X_2$ are also random.
- For any constants $C_1$ and $C_2$, the expression $C_1X_1 + C_2X_2$ is also considered a random variable.
- The absolute value of X, denoted as $|X|$, is a random variable.

## Types of Random Variable

A random variable can be classified into two distinct types.

### Discrete Random Variable

As implied by its name, this variable is not linked or continuous. It can only take on a countable number of real values, meaning the values of the discrete random sample are inherently discrete. The outcome of the random variable is contingent on chance. In simpler terms, a real-valued function defined on a discrete sample space is referred to as a discrete random variable.

Examples of discrete random variables include the number of calls a person receives in a day, the quantity of items sold by a company, the count of items manufactured, the number of accidents, and the number of gifts received on a birthday.

### Continuous Random variable

A variable that takes on an infinite range of values within the sample space is termed a continuous random variable. It has the capacity to encompass all possible values within certain limits, including both integral and fractional values. Examples of continuous random variables include height, weight, age of a person, and the distance between two cities.

## Probability Distribution (For competitive exam)

We can determine the probability associated with any event in a random experiment. Similarly, for various values of a random variable, we can ascertain their respective probabilities. The combination of the values of random variables and their corresponding probabilities constitutes the probability distribution of the random variable.

Let X be a random variable, and its probability distribution is represented by the function P(X). Any function F defined for all real x as $F(x) = P(X \leq x)$ is referred to as the distribution function of the random variable X.

## Properties of Probability Distribution (For competitive exam)

- The probability distribution of a random variable X is $P(X = x_i) = p_i$ for $x = x_i$ and $P(X = x_i) = 0$ for $x \neq x_i$.

- The probability distribution for all conceivable values of a random variable spans from 0 to 1, meaning $0 \leq p(x) \leq 1$.

## Probability Distribution of a Discrete Random Variable (For competitive exam)

If X represents a discrete random variable with distinct values $x_1, x_2, \dots, x_n, \dots$, then the probability function is denoted as $P(x) = p_X(x)$. The distribution function is given by

$$F_X(x) = P(X \leq x_i) = \sum_i p(x_i) = p_i$$

when $x = x_i$ and is 0 for other values of x. In this context, i ranges from 1 to n and beyond. To illustrate, consider the example of tossing two fair coins, where the potential outcomes are S = {HH, HT, TH, TT}. If X denotes a random variable representing the occurrence of tails, the possible values for X are 0, 1, and 2. The distribution function for X is expressed as $F(x) = P(X \leq x)$.

| Value of x | 0 | 1 | 2 |
|---|---|---|---|
| $P(X = x) = p(x)$ | $\frac{1}{4}$ | $\frac{2}{4}$ | $\frac{1}{4}$ |
| $F(x) = p(x \leq x) = \sum_i p(x_i)$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{4}{4} = 1$ |

## Probability Distribution of a Continuous Random Variable (For competitive exam)

If X is a discrete random variable with specific values $x_1, x_2, \dots, x_n, \dots$, then the probability distribution function is represented as $F(x) = p_X(x_i)$. On the other hand, for a continuous random variable

$$F_X(x) = \int p_X(x_i)\, dx$$

Where $i = 1, 2, \dots, n, \dots$

**Ex.1** Three unbiased coins are thrown. Define X as the count of heads and Y as the count of head runs, where a 'head run' signifies the consecutive occurrence of at least two heads. Determine the probability functions for X and Y.

**Sol.** The possible outcomes of the experiment is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

X represents the number of heads, assuming values of 0, 1, 2, and 3.

| EVENTS | RANDOM VARIABLES | |
|---|---|---|
| | X | Y |
| HHH | 3 | 1 |
| HHT | 2 | 1 |
| HTH | 2 | 0 |
| HTT | 1 | 0 |
| THH | 2 | 1 |
| THT | 1 | 0 |
| TTH | 1 | 0 |
| TTT | 0 | 0 |

- $P(\text{no head}) = p(0) = \frac{1}{8}$
- $P(\text{one head}) = p(1) = \frac{3}{8}$
- $P(\text{two heads}) = p(2) = \frac{3}{8}$
- $P(\text{three heads}) = p(3) = \frac{1}{8}$

| VALU OF X, | | | | |
|---|---|---|---|---|
| **x** | 0 | 1 | 2 | 3 |
| **P(x)** | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

Y is the number of head runs. It takes up the values 0 and 1.

$$P(Y = 0) = p(0) = \frac{5}{8}$$

and

$$P(Y = 1) = p(1) = \frac{3}{8}$$

| Value of Y, y | 0 | 1 |
|---|---|---|
| **P(y)** | $\frac{5}{8}$ | $\frac{3}{8}$ |

**Ex.2** Can you provide a definition for a random variable?

**Sol.** A variable is something that can alter its value and may fluctuate based on the outcomes of an experiment. If a variable's value relies on the result of a random experiment, it is categorized as a random variable, capable of assuming any real value.

**Ex.3** Can you enumerate the characteristics of a random variable?

**Sol.** A random variable exclusively assumes real values. For example, if X is a random variable and C is a constant, then CX is also recognized as a random variable. Additionally, when $X_1$ and $X_2$ are two random variables, both $X_1+X_2$ and $X_1 X_2$ are also considered random variables. Moreover, for any constants $C_1$ and $C_2$, the expression $C_1X_1 + C_2X_2$ is likewise acknowledged as a random variable. Consequently, X qualifies as a random variable.

**Ex.4** Can you provide an explanation for the concept of probability distribution?

**Sol.** We can determine the probability associated with any event in a random experiment. Similarly, for various values of the random variable, one can ascertain their respective probabilities. Furthermore, the combination of the values of random variables and their corresponding probabilities constitutes the probability distribution of the random variable.