PROBABILITY

MEAN & VARIANCE OF A RANDOM VARIABLE (FOR COMPETITIVE EXAM)

Probability and Statistics: Mean of Random Variables

If you wish to assess or speculate about your performance in five mathematics tests, each with the same total marks, you can gain insights by calculating the average of your scores. This average will provide an indication of your overall performance and indicate the marks you are closest to overall.

In the realm of probability and statistics, it is possible to determine the average of a random variable. The term "average" refers to the mean, expected value, or expectation within the context of probability and statistics. After establishing the probability distribution for a random variable, one can compute its expected value.

The mean of a random variable indicates the location or central tendency of the variable.

The expectation or mean of a discrete random variable is a weighted average of all potential values the variable can take. The weights are represented by the probabilities associated with each respective value. This is calculated as follows:

$$E(X) = \mu = \Sigma_i x_i p_i \ ; \quad i = 1, 2, \ldots, n$$

$$E(X) = x_1p_1 + x_2p_2 + \ldots + x_np_n.$$

Properties of Mean of Random Variables

- If X and Y are random variables, then the expected value of their sum equals the sum of their individual expected values:

$$E(X + Y) = E(X) + E(Y).$$

- If $X_1, X_2, \ldots, X_n$ are random variables , then the expected value of their sum is equal to the sum of their individual expected values:

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n) = \Sigma_i E(X_i)$$

- For independent random variables, X and Y, the expected value of their product equals the product of their individual expected values:

$$E(XY) = E(X)\, E(Y)$$

- If a is any constant and X is a random variable, $E[aX] = a\, E[X]$ and $E[X + a] = E[X] + a$.
- For any random variable X where $X > 0$, the expected value $E(X) > 0$.
- $E(Y) \geq E(X)$ if the random variables X and Y are such that $Y \geq X$.

Probability and Statistics: Variance of Random Variables

Imagine you've computed the mean or average marks for the five mathematics tests. By examining the marks difference in each test from this average, you can easily discern the variability in the potential values of the random variable, where the random variable represents the marks obtained in the test.

The variance of a random variable provides insight into the variability or dispersion of these random variables. It essentially indicates how far a random variable deviates from its mean. The calculation of variance involves determining the squared differences between each value and the mean.

$\sigma_x^2 = Var\,(X) = \sum_i (x_i - \mu)^2\, p(x_i) = E(X - \mu)^2$ or, $Var(X) = E(X^2) - [E(X)]^2$.

$E(X^2) = \sum_i x_i^2\, p(x_i)$, and $[E(X)]^2 = [\sum_i x_i\, p(x_i)]^2 = \mu^2$.

When the variance is low, it implies that the values of the random variable are closely clustered around the mean.

Properties of Variance of Random Variables

- The variance of any constant is zero, denoted as $V(a) = 0$, where 'a' is any constant.
- For a random variable X and constants 'a' and 'b', then
$$V(aX + b) = a^2\, V(X).$$
- For any pair-wise independent random variables, $X_1, X_2, \ldots, X_n$ and for any constants $a_1, a_2, \ldots, a_n$;
$$V(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1^2\, V(X_1) + a_2^2\, V(X_2) + \ldots + a_n^2\, V(X_n).$$

Ex.1 Compute the mean and variance for a random variable, X, defined as the count of tails in four coin tosses. Additionally, illustrate the probability distribution.

Sol. Let T represent a tail and H represent a head. The random variable X signifies the number of tails in four coin tosses, with possible values of 0, 1, 2, 3, and 4.

S. No.	Possible outcomes	Number of Tails,x	S. No.	Possible outcomes	Number of Tails,x
1	THTH	2	9	THTT	3
2	HHTH	1	10	HHTT	2
3	TTTH	3	11	TTTT	4
4	HTTH	2	12	HTTT	3
5	THHH	1	13	THHT	2
6	HHHH	0	14	HHHT	1
7	TTHH	2	15	TTHT	3
8	HTHH	1	16	HTHT	2

$$p(x=0)=\frac{1}{16}$$

$$p(x=1)=\frac{4}{16}=\frac{1}{4}$$

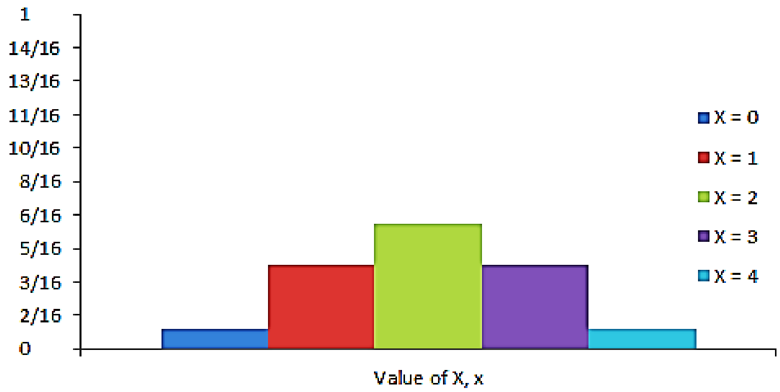
$$p(x=2)=\frac{6}{16}=\frac{3}{8}$$

$$p(x=3)=\frac{4}{16}=\frac{1}{4}$$

$$p(x=4)=\frac{1}{16}$$

The probability distribution of X is

x	0	1	2	3	4
P(x)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$



$$E(X) = \sum_i x_i p_i$$

$$= 1 \times \frac{1}{4} + 2 \times \frac{3}{8} + 3 \times \frac{1}{4} + 4 \times \frac{1}{16}$$

$$= \frac{8}{4} = 2$$

$$E(X^2) = 1^2 \times \frac{1}{4} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{4} + 4^2 \times \frac{1}{16}$$

$$= \frac{1}{4} + \frac{3}{2} + \frac{9}{4} + 1 = 5$$

So, Variance of $X = V(X)$

$$= E(X^2) - [E(X)]^2$$

$$= 5 - 2^2 = 1.$$

Ex. 2 What is the significance of probability in statistics?

Sol. Probability in statistics involves assessing the likelihood of an event occurring in a random experiment. It is expressed as a number ranging from 0 to 1, where 0 signifies impossibility, and 1 denotes certainty. The greater the probability of an event, the more likely it is to happen.

Ex.3 What is the connection between probability and statistics?

Sol. Probability is concerned with predicting the likelihood of future events, while statistics involves analyzing the frequency of past events. Probability primarily constitutes a theoretical branch of mathematics that explores the outcomes of mathematical principles.

Ex.4 What is the formula for probability?

Sol. The probability formula is expressed as:

$$\text{Probability} = \frac{(\text{Number of favorable Outcomes})}{(\text{Total number of Outcomes})}$$

$$P = \frac{n(E)}{n(S)}$$

Here, P represents the probability, E stands for the event, and S refers to the sample space.

Ex. 5 Can you provide an illustration of statistics?

Sol. Typically, we employ a statistic to estimate the value of a population parameter. For instance, the average height of the students sampled is considered a statistic. Similarly, the average grade point average would also fall into the category of a statistic. In essence, any measurable characteristic of the sample serves as an example of a statistic.