PROBABILITY

INTRODUCTION OF PROBABILITY & CONDITIONAL PROBABILITY

INTRODUCTION

In the natural world, various phenomena produce unpredictable results, like a coin toss resulting in either heads or tails. Probability theory is crafted to quantify the uncertainties linked with these outcomes. Probability offers a measure of the chance of an event taking place. It's essential to recognize that probability cannot precisely forecast the exact number of occurrences. However, the capacity to quantify the likelihood of an event is vital, as many decisions in our daily lives rely on probabilities rather than absolute certainties.

IMPORTANT DEFINITION

(A) **Experiment:** An action or process resulting in two or more distinctly defined outcomes. Examples of such activities include tossing a coin, rolling a die, or drawing a card from a thoroughly shuffled deck.

(B) **Sample Space:** A set S containing all possible outcomes of a random experiment is known as a sample space, with each individual outcome referred to as a sample point. While multiple sample spaces may describe experiment outcomes, typically, only one provides the most comprehensive information.

For example, in an experiment like "throwing a die," various sample spaces are conceivable:

(i) {even number, odd number}

(ii) {a number less than 3, a number equal to 3, a number greater than 3}

(iii) {1,2,3,4,5,6}

In this context, the third sample space is the one that offers the most comprehensive information. A sample space is considered finite if it contains a finite number of points and infinite if it comprises an infinite number of points.

For example:

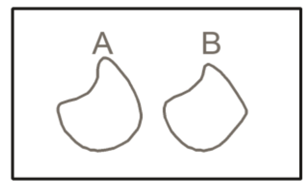
(i) In a coin toss, where the outcome can be either a head (H) or a tail (T), the sample space is $S = \{H, T\}$, making it a finite sample space.

(ii) When selecting a number from the set of natural numbers, the sample space is $S = \{1, 2, 3, 4, ...\}$, constituting an infinite sample space.

(C) **Event:** An event is a subset of the sample space. Examples include getting a head in a coin toss or obtaining a prime number when throwing a die. Generally, if a sample space comprises 'n' elements, it can be associated with a maximum of 2^n events.

(D) **Compound Event:** A compound event is identified when two or more events occur simultaneously. Symbolically, $A \cap B$ or AB signifies the concurrent occurrence of both events A and B.

(E) **Complement of an Event:** The complement of an event 'A' with respect to a sample space S is the set of all elements in S that do not belong to A. It is commonly represented as A', $\bar{A}$, or AC.



(F) **Mutually Exclusive/Disjoint/Incompatible Events:**

Two events are termed mutually exclusive if the occurrence of one precludes the possibility of the other happening simultaneously. In a Venn diagram, events A and B are shown as mutually exclusive, expressed mathematically as $A \cap B = \emptyset$. Events $A_1, A_2, A_3, \ldots, A_n$ are considered mutually exclusive events.

iff $A_i \cap A_j = \phi \, \forall i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$

(G) **Equally likely events**

If events have the same chance of occurrence, then they are considered equally likely. **For example:**

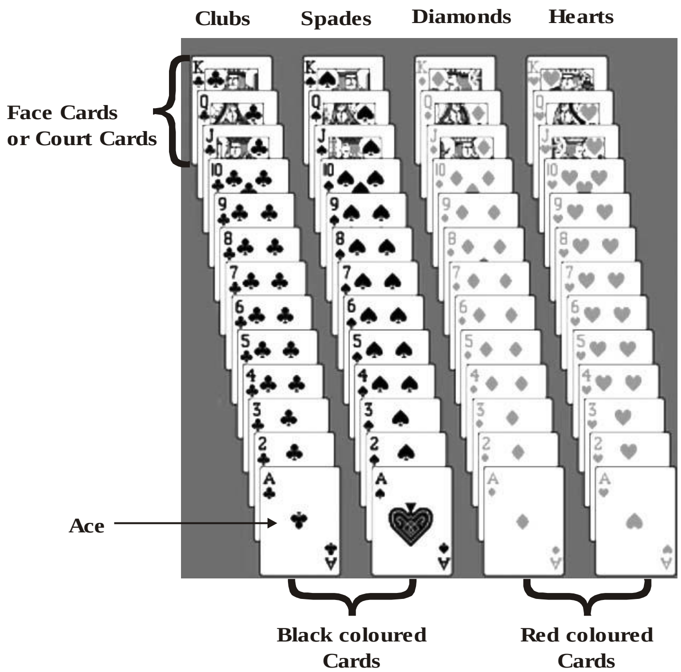
(i) In a single toss of a fair coin, the events {H} and {T} are equally likely.

(ii) In a single throw of an unbiased die the events {1}, {2}, {3} and {4}, are equally likely.

(iii) In tossing a biased coin the events {H} and {T} are not equally likely.

(H) **Exhaustive system of events**

If each outcome of an experiment is connected to at least one of the events $E_1, E_2, E_3, \ldots\ldots E_n$, then together, the events are considered exhaustive. Mathematically, we express this as $E_1 \cup E_2 \cup E_3 \ldots\ldots\ldots E_a = S.$(Sample space)



Comparative study of Equally likely, Mutually Exhaustive events:				
Experiment	**Events**	**E/L**	**M/E**	**Exhaustive**
1. Throwing of a die	A: throwing an odd face {1,3,5} B: throwing a composite {4,6}	No	Yes	No
2. A ballis drawn from an urn containing 2 White, 3 Red and 4 Green balls	E1: getting a White ball E2: getting a Red ball E3: getting a Green ball	No	Yes	Yes
3. Throwing a pair of dice	A throwing a doublet {11, 22, 33, 44, 55, 66} B: throwing a total of 10 or more { 46, 64, 55, 56, 65, 66 }	Yes	No	No
4. From a well shuffled pack of cards a card is drawn	E1: getting a heart E2: getting a spade E3: getting a diamond E4: getting a club	Yes	Yes	Yes
5. From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

Ex.1 Write the sample space for the experiment 'A coin is tossed and a die is thrown'.

Sol. The sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

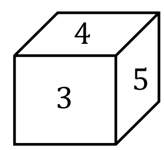
Ex.2 Determine the sample space associated with the experiment of rolling a pair of dice (plural of die) once. Also, find the number of elements in the sample space.

Sol. Let one die be blue, and the other be green. Suppose '1' appears on the blue die, and '2' appears on the green die. We denote this outcome by an ordered pair (1, 2). Similarly, if '3' appears on the blue die, and '5' appears on the green die, we denote this outcome by (3, 5), and so on. Thus, each outcome can be denoted by an ordered pair (x, y), where x is the number on the first die (blue die) and y is the number on the second die (green die). Therefore, the sample space is given by

S = {(x, y) x is the number on blue die and y is the number on green die}

We now list all the possible outcomes (figure).

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)



Ex.3 List all the events for the experiment 'tossing of a coin'.

Sol. S = {H, T}

Ex.4 A die is thrown. Let A be the event 'an odd number turns up,' and B be the event 'a number divisible by 3 turns up.' Write the events.

(A) A or B (B) A and B

Sol. A = {1, 3, 5}, B = {3, 6}

$$A \text{ or } B = A \cup B = \{1, 3, 5, 6\}$$

$$A \text{ and } B = A \cap B = \{3\}$$

Ex.5 In a single throw of a die, determine whether the events {1, 2} and {2, 3} are mutually exclusive or not.

Sol. Since $\{1, 2\} \cap \{2, 3\} = \{2\} \neq f$

the events are not mutually exclusive.

Ex.6 In the throwing of a die, let A be the event 'even number turns up,' B be the event 'an odd prime turns up,' and C be the event 'a number less than 4 turns up.' Determine whether the events A, B, and C form an exhaustive system or not.

Sol. $A \equiv \{2, 4, 6\}$,

$B \equiv \{3, 5\}$

and $C \equiv \{1, 2, 3\}$.

Clearly $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\} = S$.

Hence, the system of events covers all possibilities.

Ex.7 Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Sol. In the experiment, a head may come up on the first toss, or the second toss, or the third toss, and so on. Therefore, the desired sample space is

S = {H, TH, TTH, TTTH, TTTTH,...}

CLASSICAL DEFINITION OF PROBABILITY

If n represents the total number of equally likely, mutually exclusive, and exhaustive outcomes of an experiment, and m of them are favorable to the happening of the event A, then the probability of the event A is given by $P(A) = \frac{m}{n}$. There are (n-m) outcomes that are favorable to the event that A does not happen. The event 'A does not happen' is denoted by (A') (and is read as 'not A').

Thus $$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$$

i.e. $$P(\bar{A}) = 1 - P(A)$$

Ex. 8 When rolling a fair die, determine the probability of the event 'a number less than or equal to 4 turns up.'.

Sol. Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Event $A = \{1, 2, 3, 4\}$

$$n(A) = 4 \quad \text{and} \quad n(S) = 6$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Ex. 9 When a coin is tossed successively three times, calculate the probability of getting exactly one head or two heads.

Sol. Let S be the sample space, and E be the event of getting exactly one head or exactly two heads. Then

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}.$$

and $E = \{HHT, HTH, THH, HTT, THT, TTH\}$

$n(E) = 6$ and $n(S) = 8$.

Now required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{8} = \frac{3}{4}\ .$$

Ex.10 When rolling a pair of fair dice, determine the probability of getting a total of 8.

Sol. When a pair of dice is thrown, the sample space consists

{(1, 1) (1, 2) .... (1, 6)
(2, 1,) (2, 2,)......... (2, 6)
....
....
(6, 1), (6, 2)........ (6, 6)}

Note In this context, (1, 2) and (2, 1) are treated as distinct points to ensure each outcome is equally likely.

To obtain a total of '8', the favorable outcomes are (2, 6), (3, 5), (4, 4), (5, 3), and (6, 2).

Hence, the required probability is $\frac{5}{36}$

Ex.11 Words are created using the letters of the word PEACE. Determine the probability that two E's come together.

Sol. The total number of words that can be formed with the letters P, E, A, C, E $= \frac{5!}{2!} = 60$

The number of words in which two E's come together = 4! = 24

Required probability = $\frac{24}{60} = \frac{2}{5}$

Ex.12 A bag holds 5 red and 4 green balls. When four balls are randomly drawn, determine the probability of getting two red balls and two green balls.

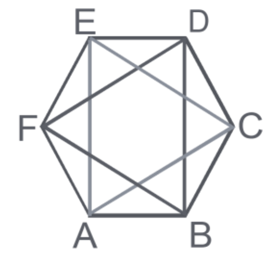
Sol. n(s) represents the total number of ways of drawing 4 balls out of total 9 balls: ${}^9C_4$

Event A is defined as drawing 2 red and 2 green balls; $n(A) = {}^5C_2 \times {}^4C_2$

$$P(A) = \frac{n(A)}{n(s)} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{\frac{5\times4\times4\times3}{2\times2}}{\frac{9\times8\times7\times6}{4\times3\times2}} = \frac{10}{21}$$

Ex.13 Selecting three vertices randomly from a regular hexagon with six vertices, the probability of forming an equilateral triangle by joining the chosen vertices is -

Sol. The total number of cases $= {}^{6}C_{3} = 20$



As illustrated in the figure, only two triangles ACE and BDF are equilateral.

Therefore, the number of favorable cases is 2.

Thus, the required probability $= \frac{2}{20} = \frac{1}{10}$