PROBABILITY

BERNOULLI TRIALS AND BINOMIAL DISTRIBUTION (FOR COMPETITIVE EXAM)

BERNOULLI TRIAL

Named after Jacob Bernoulli, the Bernoulli trials process is a fundamental and crucial random process in probability. It essentially serves as a mathematical abstraction of coin tossing, and due to its broad applicability, it is often described in the context of a series of generic trials. A sequence of Bernoulli trials adheres to the following assumptions:

(A) Each trial presents two potential outcomes, referred to as success and failure in reliability terminology.

(B) The trials are independent, meaning the result of one trial does not influence the outcome of another trial.

(C) In each trial, the probability of success is denoted as p, while the probability of failure is 1-p, where p lies within the range [0, 1] and represents the success parameter of the process.

In many real-life scenarios, events often boil down to just two consequential outcomes. For instance, passing or failing an exam, securing or not securing a job, experiencing a flight delay or having an on-time departure are all situations where only two outcomes matter. The probability theory abstraction for such situations is encapsulated in a Bernoulli trial.

A Bernoulli trial is an experiment characterized by only two potential outcomes, both having positive probabilities denoted as p and q, where $p + q = 1$. These outcomes are typically labeled as "success" and "failure," often represented by "S" and "F," or alternatively as 1 and 0.

For instance, when rolling a die, our focus might be solely on the outcome of getting 1, in which case, naturally, $P(S) = \frac{1}{6}$ and $P(F) = \frac{5}{6}$. If, when rolling two dice, our sole interest lies in whether the sum on both dice is 11, then $P(S) = \frac{1}{18}$, $P(F) = \frac{17}{18}$.

BINOMIAL PROBABILITY THEOREM

Consider p as the probability of success in a single Bernoulli trial. Then, $q = 1 - p$ represents the probability of failure in a single trial. The likelihood that the event will occur precisely x times in n trials (meaning there are x successes and $n - x$ failures) is determined by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad \text{...... (i)}$$

Here, the random variable X represents the count of successes in n trials, and $x = 0, 1, \ldots\ldots, n$.

Ex.1 The probability of obtaining precisely 2 heads in 6 tosses of a fair coin is.

Sol.

$$P(X = 2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{6!}{2!4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{15}{64}$$

The discrete probability function, often denoted as (i), is commonly referred to as the binomial distribution. This is because, for $x = 0, 1, 2, \ldots, n$, it aligns with successive terms in the binomial expansion.

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \ldots + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The Bernoulli distribution is a specific instance of the binomial distribution when $n = 1$.

Ex.2 A pair of dice is thrown 5 times. Determine the probability of getting a doublet twice.

Sol. In a single throw of a pair of dice, the probability of getting a doublet is $\frac{1}{6}$

represented as $p = \frac{1}{6}$

$$q = 1 - \frac{1}{6} = \frac{5}{6}$$

number of success r = 2

$$P(r = 2) = {}^3C_2 p^2 q^3$$

$$= 10 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$$

Ex.3 In a 5-match hockey test series between India and Pakistan, the probability of India winning at least three matches is -

Sol. India wins at least three matches

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5 (16) = \frac{1}{2}$$

Ex.4 In an examination consisting of 10 multiple-choice questions (each having 4 options where one or more can be correct), a student chooses to answer randomly. Determine the probability of the student getting exactly two questions correct.

Sol. A student has the possibility of marking 15 different answers to a multiple-choice question with 4 options.

i.e. $${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$$

Therefore, if he randomly selects an answer, the probability of his answer being correct is $\frac{1}{15}$ and the likelihood of being incorrect is $\frac{14}{15}$.

Thus, $$p = \frac{1}{15}, q = \frac{14}{15}$$

$$P(2\text{ success}) = {}^{10}C_2 \times \left(\frac{1}{15}\right)^2 \times \left(\frac{14}{15}\right)^8$$

Ex.5 A man moves forward with a probability of 0.4 and backward with a probability of 0.6. Determine the probability that, at the end of eleven steps, he is one step away from the starting point.

Sol. If the man is one step away from the starting point, it means either

(i) He has taken 6 steps forward and 5 steps backward, or

(ii) He has taken 5 steps forward and 6 steps backward.

Considering each step forward as success and each step backward as failure.

Let p be the probability of success (1 step forward) = 0.4,

and q be the probability of failure (1 step backward) = 0.6.

$$\text{Required Probability} = P\{X = 6 \text{ or } X = 5\}$$

$$= P(X = 6) + P(X = 5)$$

$$= {}^{11}C_5 p^6 q^5 + {}^{11}C_s p^5 q^6$$

$$= {}^{11}C_5 \left(p^6 q^5 + p^5 q^6\right)$$

$$= \frac{11.10.9.8.7}{11^2 5}\left\{(0.4)^6 (0.6)^5 + (0.4)^5 (0.6)^6\right\}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8.7}{1 \cdot 2 \cdot 3 \cdot 4.5}(0.24)^5$$

Hence the required probability = 0.37