THREE DIMENSIONAL GEOMETRY

SHORTEST DISTANCE BETWEEN TWO LINES

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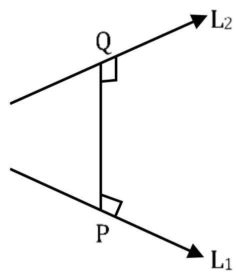
DEFINITION

(i) **Line of shortest distance:** If L_1 and L_2 are two skew lines, there exists a unique line that is perpendicular to both, and it is referred to as the line of shortest distance.

(ii) **Shortest distance:** The shortest distance between two lines, L_1 and L_2, is represented by the distance |PQ|, where P and Q are points at which the line of shortest distance intersects L_1 and L_2, respectively.



Note: If two lines in space intersect at a point, then the minimum distance between them is zero.

(a) **To find the shortest distance between two skew line $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$.**

Let two skew line L_1 and L_2 be the $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ ... (1)

And $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ ... (2)

Take any point S $(\vec{a}_1)$ on L_1 and T $(\vec{a}_2)$ on L_2.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

By def. $\overrightarrow{PQ}$ is perp. To (1) and (2)

$\overrightarrow{PQ}$ is prep. To both $\vec{b}_1$ and $\vec{b}_2$.

$\overrightarrow{PQ}$ is parallel to $\vec{b}_1 \times \vec{b}_2$.

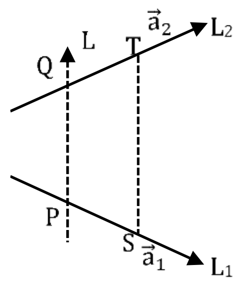
The unit vector $\hat{n}$ along PQ is given by $\hat{n} = \dfrac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|}$

Let $\overrightarrow{PQ} = d\,\hat{n}$., where d is the magnitude of the vector representing the shortest distance.

Clearly PQ is a projection of $\overrightarrow{ST}$ on $\overrightarrow{PQ}$

Now if 'θ' be the angel between $\overrightarrow{PQ}$ and $\overrightarrow{ST}$.

Then $$PQ = ST\cos\theta.$$



But $$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{ST}}{|\overrightarrow{PQ}||\overrightarrow{ST}|} = \frac{d\hat{n}\cdot\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{d\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}$$

Hence, $$d = PQ = ST\cos\theta = \frac{\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)\cdot\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}$$

Since d is always considered positive.

$$d = \left|\frac{\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)\times\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}\right|$$

Cor. If two line intersect then d = 0

i.e. $$\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)\cdot\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right) = 0$$

(b) To find the shortest distance between two parallel lines:

$$\vec{r} = \overrightarrow{a_1} + \lambda\vec{b} \text{ and } \vec{r} = \overrightarrow{a_2} + \mu\vec{b}.$$

Let two parallel line L_1 and L_2 be:

$$\vec{r} = \overrightarrow{a_1} + \lambda\vec{b} \quad \text{... (1)}$$

$$\vec{r} = \overrightarrow{a_2} + \lambda\vec{b} \quad \text{... (2)}$$

These are clearly coplanar.

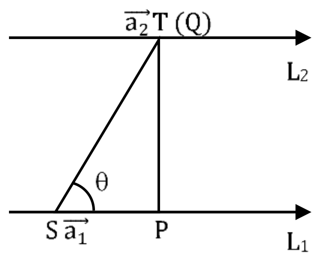
It is evident that either L_1 and L_2 is parallel to vector $\vec{b}$, and both lines pass through the points S $(\overrightarrow{a_1})$ and T$(\overrightarrow{a_2})$.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

$$d = PQ = ST\cos(90°-\theta) = ST\sin\theta$$
$$= (ST)\frac{|\vec{b}\times\overrightarrow{ST}|}{|\vec{b}|(ST)} = \frac{\vec{b}\times\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{|\vec{b}|}.$$

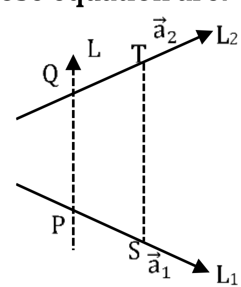
Since d is always to be taken as a positive.

$$d = \left|\frac{\vec{b}\times\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{|\vec{b}|}\right|.$$



(C) To find the shortest distance between two straight lines whose equation are:

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$
$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$



Let PQ be the S.D.

Let <l, m, n,> be its direction cosines.

Then $ll_1 + mm_1 + nn_1 = 0$ ... (1)

And $ll_2 + mm_2 + nn_2 = 0$ ... (2)

Solving, $$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}.$$

Direction ratios of PQ are:

$$\langle m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1\rangle.$$

Direction cosines of PQ are :

$$< \frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma\left(m_1n_2 - m_2n_1\right)^2}}, \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma\left(m_1n_2 - m_2n_1\right)^2}}, \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma\left(m_1n_2 - m_2n_1\right)^2}} >.$$

Length of the S.D. = |PQ| = Projection of |AB| on PQ

$$\frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}.$$

Cor. **If two line intersect then:**

$$(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)=0$$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

CO-PLANARITY OF TWO LINES:

Consider the two lines:

$$\vec{r}=\overrightarrow{a_1}+\lambda\overrightarrow{b_1} \qquad \dots (1)$$
$$\vec{r}=\overrightarrow{a_2}+\mu\overrightarrow{b_2} \qquad \dots (2)$$

1. Passes through point A with the position vector $\overrightarrow{a_1}$ and is parallel to $\overrightarrow{b_1}$.
2. Passes through point B with the position vector $\overrightarrow{a_2}$ and is parallel to $\overrightarrow{b_2}$.

$$\overrightarrow{AB}= \text{ p.v. of B} - \text{ p.v. of A} = \overrightarrow{a_2}-\overrightarrow{a_1}.$$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is prep. To $\overrightarrow{b_1}\times\overrightarrow{b_2}$

$$\overrightarrow{AB}\cdot\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)=0$$

$$\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)\cdot\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)=0$$

d, the shortest distance between (1) and (2) = 0.

CARTESIAN FORM:

Let the coordinates of points A and B be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ have direction ratios:

$\langle a_1, b_1, c_1\rangle$ and $\langle a_2, b_2, c_2\rangle$ respectively.

$$\overrightarrow{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$
$$\overrightarrow{b_1'}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$$
$$\overrightarrow{b_2'}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$$

These given lines are coplanar if and only if

$$\overrightarrow{AB}\cdot\left(\overrightarrow{b_1'}\times\overrightarrow{b_2'}\right)=0$$

This can be written in Cartesian form as:

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}-0$$

Ex.1 The vector equations of the two lines are:

$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$$

$$\vec{r}=2\hat{i}+4\hat{j}+5\hat{k}+\mu(3\hat{i}+4\hat{j}+5\hat{k})$$

Determine the shortest distance between these lines.

Sol. Comparing the provided equation with:

$$\vec{r}=\overrightarrow{a_1}+\lambda\overrightarrow{b_1}$$

$$\vec{r}=\overrightarrow{a_2}+\mu\overrightarrow{b_2},$$

We have:

$$\overrightarrow{b_1}=2\hat{i}+3\hat{j}+4\hat{k}$$

$$\overrightarrow{b_3}=3\hat{i}+4\hat{j}+5\hat{k}$$

$$\overrightarrow{a_1}=\hat{i}+2\hat{j}+3\hat{k},$$

$$\overrightarrow{a_2}=2\hat{i}+4\hat{j}+5\hat{k}$$

$$\overrightarrow{b_1}\times\overrightarrow{b_2}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$\hat{i}(15-16)-\hat{j}(10-12)+\hat{k}(8-9)$$

$$-\hat{i}+2\hat{j}-\hat{k}$$

Ex.2 Determine the distance between the lines L_1 and L_2 represented by:

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})\text{ and}$$

$$\vec{r}=3\hat{i}+3\hat{i}-5\hat{k}+u(2\hat{i}+3\hat{i}+6\hat{k}).$$

Sol. Clearly L_1 and L_2 are parallel.

Comparing given equation with:

$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \overrightarrow{a_2} + \mu\vec{b}$,

We have: $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$|\vec{b}| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\overrightarrow{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}; \overrightarrow{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = (3-1)\hat{i} + (3-2)\hat{j} + (-5+4)\hat{k}$$

$$2\hat{i} + \hat{j} - \hat{k}$$

$$\vec{b} \times \left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\hat{i}(-3-6) - \hat{j}(-2-12) + \hat{k}(2-6)$$

The distance, denoted as d, between the provided lines is given by:

$$d = \left| \frac{\vec{b} \times \left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{|\vec{b}|} \right|$$

$$= \left| \frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{7} \right|$$

$$= \frac{1}{7} |-9\hat{i} + 14\hat{j} - 4\hat{k}|$$

$$= \frac{1}{7}\sqrt{81+196+16}$$

$$= \frac{1}{7}\sqrt{293} \text{ units.}$$

Ex.3 Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect.

Also, determine the point of intersection of these lines.

Sol. The given lines are:

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \qquad \dots (1)$$

$$\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(-1)^2 + (2)^2 + (-1)^2}$$

$$= \sqrt{1+4+1} = \sqrt{6}.$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = (2-1)\hat{i} + (4-2)\hat{j} + (5-3)\hat{k}$$

$$= \hat{i} + 2\hat{j} + 2\hat{k}$$

d the shortest distance between the given line is given by:

$$d = \left| \frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right) \cdot \left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|} \right|$$

$$= \left| \frac{(-\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{6}} \right|$$

$$= \left| \frac{(-1)(1) + (2)(2) + (-1)(2)}{\sqrt{6}} \right|$$

$$= \left| \frac{-1+4-2}{\sqrt{6}} \right| = \left| \frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \text{ units.}$$

$$L_2 : \frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} \qquad \ldots (2)$$

Any point on L_1 is $(2\lambda+1, 3\lambda+2, 4\lambda+3)$ ... (3)

Any point on L_2 is $(5\mu+4, 2\mu+1, \mu)$ ... (4)

The line L_1 and L_2 will increase iff point (3) and (4) coincide.

Iff $2\lambda+1 = 5\mu+4, 3\lambda+2 = 2\mu+1, 4\lambda+3 = \mu$

Taking first two. $2\lambda - 5\mu = 3$ ... (5)

Taking middle two. $3\lambda - 2\mu = -1$ ... (6)

Taking last two. $4\lambda - \mu = -3$ ... (7)

Solving (5) and (6) $\lambda = -1$ and $\mu = -1$.

Putting in $(7) 4(-1) + 1 = -3$

$\Rightarrow -3 = -3$, Which is true.

Hence the given line L_1 and L_2 intersect.

Putting $\lambda = -1$ in (3), [or $\mu = -1$ in (4)],

We get the reqd. point of intersection as. $(-1, -1, -1)$.

Ex.4 Determine whether the line:

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j})$$

$$\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

Determine if they intersect. If they do, find their point of intersection.

Sol. The given line are:

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j})$$

$$\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

$$\vec{r} = (1 + 2\lambda)\hat{i} + (-1 + \lambda)\hat{j} - \hat{k} \qquad \dots (1)$$

$$\vec{r} = (2 + \mu)\hat{i} + (-1 + \mu)\hat{j} - \mu\hat{k} \qquad \dots (2)$$

If the line (1) and (2) intersect, then for some value of λ and.μ, we have:

$$1 + 2\lambda = 2 + \mu \qquad \dots (3)$$

$$-1 + \lambda = -1 + \mu \qquad \dots (4)$$

$$-1 - -u \Rightarrow u = 1 \qquad \dots (5)$$

Putting in (4), $-1 + \lambda = -1 + 1 \Rightarrow \lambda = 1.$

Thus, $\lambda = 1$ and $\mu = 1$

These also satisfy (3) $[\because 1 + 2(1) = 2 + 1 \text{ i.e.} 3 = 3]$

Hence the line intersect.

Putting $\lambda = 1$ in (1).

$$\vec{r} = (1 + 2)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}$$

Putting $\mu = 1$ in (2).

$$\vec{r} = (2 + 1)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}$$

Hence the point intersection is (3, 0, – 1)

Ex.5 Determine the shortest distance and the vector equation of the line representing the shortest distance between the given lines.

$$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$$

Sol. The provided equation in Cartesian form is:

$$L_1 : \frac{x - 3}{3} = \frac{y - 8}{-1} = \frac{z - 3}{1} (= \lambda) \qquad \dots (1)$$

$$L_2 : \frac{x + 3}{-3} = \frac{y + 7}{2} = \frac{z - 6}{4} (= \mu) \qquad \dots (2)$$

Any point on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$.

Any point L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$.

If the line of shortest distance intersection (1) in P and (2) in Q. then the direction-ratios of $\overrightarrow{PQ}$ are.

$$< -3\mu - 3 - 3\lambda - 3, 2\mu - 7 + \lambda - 8, 4\mu + 6 - \lambda - 3 >$$

$$< 3\lambda + 3\mu + 6 \cdot -\lambda - 2\mu + 15 \cdot \lambda - 4\mu - 3 >$$

Since PQ is prep. To the line (1).

$$(3)\,(3\lambda+3\mu+6)+(-1)(-\lambda-2\mu+15)\ +(1)(\lambda-4\mu-3)=0$$

$$11\lambda+7\mu=0 \qquad \ldots (3)$$

Since PQ is prep, to line (2).

$$(-3)(3\lambda+3\mu+6)+2(-\lambda-2\mu+15)\ +4(\lambda-4\mu-3)=0$$

$$-7\lambda-29\mu=0 \qquad \ldots (4)$$

Solving (3) and (4), $\lambda = 0$ and $\mu = 0$.

Point P and Q are (3, 8, 3) and (–3, –7, 6) respectively.

$$\text{S.D.} = |PQ|$$

$$=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}$$

$$=\sqrt{36+225+9}=\sqrt{270}=3\sqrt{30}\ \text{units}$$

And the equation of line S.D. is:

$$\vec{r}=(3\hat{i}+8\hat{j}+3\hat{k})+\mu(-6\hat{i}-15\hat{j}+3\hat{k})$$

$$[\text{Using}\ \vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})\ ]$$