THREE DIMENSIONAL GEOMETRY

PLANE (FOR COMPETITIVE EXAM)

INTRODUCTION

A plane is a surface in which, if we select any two distinct points on it, the line joining these points entirely lies within the plane.

We can define a specific plane in various ways, such as:

1. A unique plane can be determined by drawing it through three non-collinear points. Therefore, three given non-collinear points specify a particular plane.
2. A single plane can be drawn to contain two concurrent lines. Thus, two given concurrent lines specify a particular plane.
3. Only one plane can be drawn perpendicular to a given direction at a specified distance from the origin. Hence, the normal to the plane and the distance from the origin specify a particular plane.
4. Only one plane can be drawn through a given point and perpendicular to a given direction. Thus, a point on the plane and a normal to the plane specify a particular plane. Of these, options 3 and 4 are the most useful.

EQUATION OF PLANE:

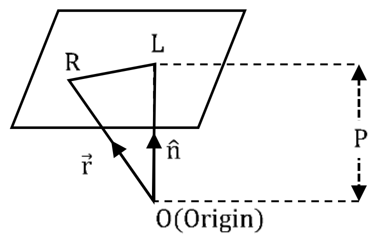
(a) Standard Form (Normal Form)

To determine the equation of a plane at a distance 'p' from the origin and perpendicular to the unit vector $\hat{n}$ (directed away from O), consider O as the origin, 'p' as the length of the perpendicular CL from O to the given plane, and $\hat{n}$ as the unit vector normal to the plane in the direction away from O, i.e., from O to L.

Then $\overrightarrow{OL} = p\hat{n}$

Let p be any point on the plane whose position vector is $\vec{r}$

Clearly $\overrightarrow{LP}$ is $\perp \overrightarrow{OL}$



P is a point on the plane

$$\overrightarrow{LP} \cdot \overrightarrow{OL} = 0 \qquad \text{... (A)} \quad \text{(Scaler – product form)}$$

Now $\overrightarrow{LP} = \overrightarrow{OP} - \overrightarrow{OL} = \vec{r} - p\hat{n}$

From (A) $(\vec{r} - p\hat{n}) \cdot \hat{n} = 0$

$$\vec{r} \cdot \hat{n} - p\hat{n} \cdot \hat{n} = 0$$

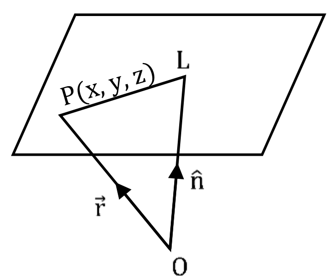
$$\vec{r} \cdot \hat{n} - p = 0 \qquad (\because \hat{n} \cdot \hat{n} = I)$$

$$\vec{r} \times \hat{n} = P \qquad \text{... (B)} \quad \text{(Standard form)}$$

Cartesian Form:

Let $\vec{r}.\hat{n} = P$ ... (1)

Be the vector equation of the plane where $\hat{n}$ is the unit vector normal to the plane as shown in the figure.



Consider any point P(x, y, z) on the plane, then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If < l, m, n> are direction – cosines of $\hat{n}$ then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

Putting in (1), we get: $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$

$$lx + my + nz = p.$$

Therefore, <l, m, n> represents the direction cosines of the normal to the plane, and 'p' is the length of the perpendicular from the origin to the plane.

(b) GENERAL FORM

To demonstrate that the general equation of the first degree in x, y, z represents a plane, consider the general equation as $ax + by + cz + d = 0$.

Where a, b, c, are not all zero

$$a^2 + b^2 + c^2 \neq 0.$$

$$ax + by + cz = -d$$

If P is not in the plane $\overrightarrow{LP}$ is not perp. to $\overrightarrow{OL}$.

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(a\hat{i}+b\hat{j}+c\hat{k})=-d$$

$$\vec{r}\cdot\left\{\frac{a}{\sqrt{a^2+b^2+c^2}}\hat{i}+\frac{b}{\sqrt{a^2+b^2+c^2}}\hat{j}+\frac{c}{\sqrt{a^2+b^2+c^2}}\hat{k}\right\}=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$$\vec{r}\cdot(l\hat{i}+m\hat{j}+n\hat{k})=p,\text{ where }\frac{-d}{\sqrt{a^2+b^2+c^2}}=p$$

$\vec{r}\cdot\hat{n}=p$, where $\hat{n}=l\hat{i}+m\hat{j}+n\hat{k}$ Which represent a plane.

Hence, ax + by + cz +d = 0 represent a plane the length of whose perpendicular form the origin is $p=\frac{-d}{\sqrt{a^2+b^2+c^2}}$ and the direction – ratios of the normal are <a, b, c>.

Cor. Reduction to normal Form.

$$ax+by+cz+d=0$$

$$-ax-by-cz=d$$

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}x+\frac{-b}{\sqrt{a^2+b^2+c^2}}y+\frac{-c}{\sqrt{a^2+b^2+c^2}}z=\frac{d}{\sqrt{a^2+b^2+c^2}}$$

$$\left[\text{Dividing by}\sqrt{(-a)^2+(-b)^2+(-c)^2}\right]$$

$$lx+my+nz=p,$$

$$l\left(=\frac{-a}{\sqrt{a^2+b^2+c^2}}\right),m\left(=\frac{-b}{\sqrt{a^2+b^2+c^2}}\right),n\left(=\frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$$

Represent the direction cosines of the normal to the plane and $p\left(=\frac{d}{\sqrt{a^2+b^2+c^2}}\right)$ is the length of the perpendicular form the origin to the plane.

Note: In the normal form of the plane equation lx + my + nz = P, the following characteristics should be observed:

1. P is always positive
2. (coeff. of x) ^{2} + (coeff. of y) ^{2} + (coeff. of z)^2 = I.

Another Form: The equation of the plane in normal form can also be expressed as:

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

$$I = \cos\alpha,\ m = \cos\beta.\ n = \cos \gamma$$

(c) ONE-POINT FORM

To demonstrate that the vector equation of a plane passing through a given point $\overrightarrow{r_1}$ and perpendicular to a given vector $\hat{n}$ is

$$(\vec{r} - \vec{r}_1) \cdot \hat{n} = 0$$

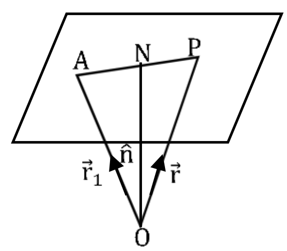
Let O be the origin let $\vec{r}_1$ be the position vector of a point A. lying on the plane. Let P be any point on the plane whose position vector is $\vec{r}$.

$\therefore$ $$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{r}_1$$

As $\overrightarrow{AP}$ lies in the plane and $\hat{n}$ is normal to the plane.

$$(\vec{r} - \vec{r}_1) \cdot \hat{n} = 0,$$

Which is the reqd. equation.



Cartesian Form: Equation of a plane passing through (x_1, y_1, z_1) and normal to the plane with direction ratios <a, b, c>.

Let. $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}; \vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and } \hat{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

Then $$\left(\vec{r} - \vec{r}_1\right) \cdot \hat{n} = 0$$

Becomes, $$\left(\vec{r} - \vec{r}_1\right) \cdot \frac{\vec{n}}{|\vec{n}|} = 0$$

$$\left[\left(x - x_1\right)\hat{i} + \left(y - y_1\right)\hat{j} + \left(z - z_1\right)\hat{k}\right] \times (a\hat{i} + b\hat{j} + c\hat{k}) = 0$$

$$a\left(x - x_1\right) + b\left(y - y_1\right) + c\left(z - z_1\right) = 0$$

(d) Equation of a plane through a point and parallel to two given lines.

To determine the equation of a plane passing through a point with position vector $\vec{a}$ and parallel to the lines.

$$\vec{r} = \vec{a} + \lambda\vec{b} \text{ and } \vec{r} = \vec{a'} + \mu\vec{b'}$$

Consider the point A, represented by the position vector $\vec{a}$

Consider P be any point in the phase whose position vector $\vec{r}$

Now P lies in the plane iff $\overrightarrow{AP}$. $\vec{b}$ And $\vec{b}$ are coplanar.

Iff $$\overrightarrow{AP}\cdot\left(\vec{b}\times\vec{b'}\right) = 0$$

Iff $$(\vec{r}-\vec{a})\cdot\left(\vec{b}\times\vec{b'}\right) = 0 \qquad \dots (1)$$

This is the required vector equation of the plane.

Cartesian Form.

Let $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{b'} = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$$

Putting in (1), we get:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \end{vmatrix} = 0.$$

This is the required Cartesian equation of the plane.

Ex.1 Determine the vector and Cartesian equations of the plane passing through the point (1, 2, –4) and parallel to the given line.

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{r} = \hat{i} - 3\hat{j} + 5\hat{k} + \mu(\hat{i} + \hat{j} - \hat{k}).$$

Sol. We have, $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{b}' = \hat{i} + \hat{j} - \hat{k}$

(i) The equation representing the plane in vector form is:

$$(\vec{r}-\vec{a})\cdot\left(\vec{b}\times\vec{b'}\right) = 0$$

$$[\vec{r} - (\hat{i} + 2\hat{j} - 4\hat{k})]\cdot[(2\hat{i} + 3\hat{j} + 6\hat{k})\times(\hat{i} + \hat{j} - \hat{k})] = 0$$

(ii) The equation representing the plane in Cartesian form is:

$$\begin{vmatrix} x-1 & y-2 & z+4 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$(x-1)(-3-6)-(y-2)(-2-6)+(z+4)(2-3)=0$$

$$-9x+9+8y-16-z-4=0$$

$$9x-8y+z+11=0$$

(e) Equation of a plane containing two lines.

To find the equation of a plane through two lines:

$$\vec{r}=\vec{a}+\lambda\vec{b} \text{ and } \vec{r}=\overrightarrow{a'}+\mu\overrightarrow{b'}$$

The given lines are $\vec{r}=\vec{a}+\lambda\vec{b}$ .....(1)

And $\vec{r}=\vec{a}'+\mu\vec{b}'$ ......(2)

(1) is a line through A($\vec{a}$) and parallel to vector $\vec{b}$.

(2)is a line trough A' ($\vec{a}'$) and parallel to vector $\vec{b}$

Let P ($\vec{r}$) be any point in the plane.

Now P lies in the plane iff $\overrightarrow{AP}, \vec{b}$ and $\vec{b}'$ are coplanar.

iff $(\vec{r}-\vec{a})\cdot\left(\vec{b}\times\vec{b}'\right)=0$ .....(3)

Which is the reqd. vector equation of the plane.

Cartesian Form.

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}, \vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

And $\vec{b}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}, \overrightarrow{b'}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

Putting in (3), we get: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$

Ex.2 Determine the vector and Cartesian forms of the equation of the plane that contains two lines:

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(-2\hat{i}+3\hat{j}+8\hat{k})$$

Sol. We have: $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{b'} = -2\hat{i} + 3\hat{j} + 8\hat{k}$

(1) The equation representing the plane in vector form is:

$$[\vec{r} - (\hat{i} + 2\hat{j} - 4\hat{k})] \cdot [(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (-2\hat{i} + 3\hat{j} + 8\hat{k})] = 0$$

(2) The Cartesian equation of the plane is.

$$\begin{vmatrix} x-1 & y-2 & z+4 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix} = 0$$

$$(x-1)(24-18) - (y-2)(16+12) + (z+4)(6+6) = 0$$

$$6x - 6 - 28y + 56 + 12z + 48 = 0$$

$$6x - 28y + 12z + 98 = 0$$

$$3x - 14y + 6z + 49 = 0.$$

(f) THREE – POINT FORM

Equation of a plane passing through three points.

To find the equation of a plane through three points having position

Vector, $\vec{r_1}, \vec{r_2}$ and $\vec{r_3}$

Consider three given points A, B, C with position vectors $\vec{r_1}, \vec{r_2}$and $\vec{r_3}$ and $\vec{r_3}$ respectively. Let $\vec{r}$ be the position vector of any point P on the plane.

Then $\overrightarrow{AP}, \overrightarrow{BP}$ and $\overrightarrow{CP}$ are coplanar vector

$$\overrightarrow{AP} \cdot (\overrightarrow{BP} \times \overrightarrow{CP}) = 0$$

$$(\vec{r} - \vec{r_1}) \cdot \left[(\vec{r} - \vec{r_2}) \times (\vec{r} - \vec{r_3}) \right] = 0.$$

Cartesian Form: Equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}, \vec{r_3} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$

$$(\vec{r} - \vec{r_1}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}, (\vec{r} - \vec{r_2}) = (x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k}$$

$$(\vec{r} - \vec{r_3}) = (x - x_3)\hat{i} + (y - y_3)\hat{j} + (z - z_3)\hat{k}$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0$$

Note. In a numerical example, to determine the equation of a plane passing through three given points, it is more concise to proceed.

(g) Equation of a plane through two given point and parallel to a given vector.

To demonstrate that the equation of the plane through two points A and B with position vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ respectively, and parallel to a given vector $\overrightarrow{m}$ is

$$(\vec{r}-\overrightarrow{r_1}).[(\overrightarrow{r_2}-\overrightarrow{r_1})\times\overrightarrow{m}]=0$$

Since the point A and B lie in the plane.

$\overrightarrow{AB}$ is parallel to the plane

$\vec{r}_2-\vec{r}_1$is parallel to the plane.

But $\overrightarrow{m}$ is parallel to the plane.

$(\overrightarrow{r_2}-\overrightarrow{r_1})\times\overrightarrow{m}$ Is normal to the plane.

The equation of the plane through $\overrightarrow{r_1}$ with $(\overrightarrow{r_2}-\overrightarrow{r_1})\times\overrightarrow{m}$ as the normal is $(\vec{r}-\overrightarrow{r_1})$.

$$[(\overrightarrow{r_2}-\overrightarrow{r_1})\times\overrightarrow{m}]=0$$

Which is the reqd. equation of the plane.

Cartesian Form: Equation of a plane passing through the points x_1, y_1, z_1 and x_2, y_2, y_1 and parallel to the line with direction ratios <a, b, c>.

Let. $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k},\overrightarrow{r_1}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k},\overrightarrow{r_2}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$

Then $r-\vec{r}_1=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$

And $\overrightarrow{r_2}-\overrightarrow{r_1}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{j}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$.

Then the equation of the plane is $(\vec{r}-\vec{r}_1)\cdot\left[\left(\overrightarrow{r_2}-\overrightarrow{r_1}\right)\times\vec{m}\right]=0$

$$\left[\vec{r}-\overrightarrow{n_1}\overrightarrow{r_2}-\overrightarrow{n_1}\vec{m}\right]=0$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0$$

(h) INTERCEPT FORM

To determine the equation of the plane that intersects the coordinate axes at points a, b, c.

Consider the desired plane (not passing through the origin O) intersecting the coordinate axes at points A, B, C.

Then OA = a, OB = b, and OC = c.

The co-ordinate of A, B and C are (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.

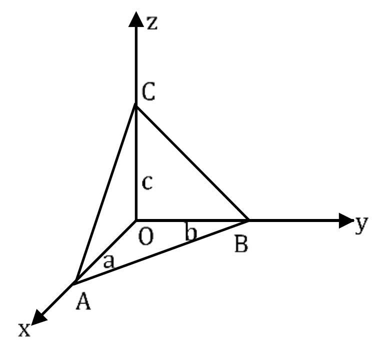
Let the equation of the required plane be A'x +B'y +C'z – D' = 0 ... (1)

Since (1) passes through the point A (a, 0, 0), B (0, b, 0) and C (0, 0, c)

$$A'(a) + D' = 0 \quad \Rightarrow \quad A' = -\frac{D'}{a}$$

$$B'(b) + D' = 0 \quad \Rightarrow \quad B' = -\frac{D'}{b}$$

$$C'(c) + D' = 0 \quad \Rightarrow \quad C' = -\frac{D'}{c}.$$



Putting these value in (1) we get

$$-\frac{D'}{a}x - \frac{D'}{b}y - \frac{D'}{c}z + D' = 0$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

Which is the required equation.