THREE DIMENSIONAL GEOMETRY

ANGEL BETWEEN TWO LINES

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(a) To determine the angle between two lines, L_1 and L_2, characterized by direction cosines $< l_1, m_1, n_1>$ and $<l_2, m_2, n_2 >$ respectively.

Let's take into account lines PQ (referred to as L_1) and ST (referred to as L_2) with direction cosines denoted by $<l_1, m_1, n_1>$ and $<l_2, m_2, n_2>$ respectively.

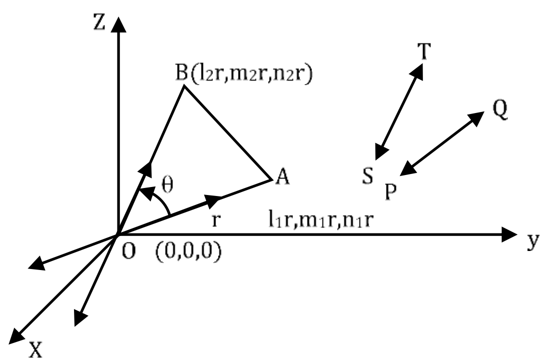
Let 'θ' represent the angle between these lines.

Through O draw OA (= r) || to PQ and OB (= r) || to ST.

Thus $\angle AOB = \theta$.

[Since the angle between two lines equals the angle between their parallels]

The coordinates of points A and B are (l_1r, m_1r, n_1r) and (l_2r, m_2r, n_2r) respectively.



$$|AB| = \sqrt{(l_2r - l_1r)^2 + (m_2r - m_1r)^2 + (n_2r - n_1r)^2}$$

$$= r\sqrt{(l_2 - l_1)^2 + (m_2 - m_1)^2 + (n_2 - n_1)^2}$$

$$= r\sqrt{l_2^2 + l_1^2 - 2l_2l_1 + m_2^2 + m_1^2 - 2m_2m_1 + n_2^2 + n_1^2 - 2n_2n_1}$$

$$= r\sqrt{(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) - 2(l_1l_2 + m_1m_2 + n_1n_2)}$$

$$= r\sqrt{1 + 1 - 2(l_1l_2 + m_1m_2 + n_1n_2)}$$

$$= r\sqrt{2 - 2(l_1l_2 + m_1m_2 + n_1n_2)}$$

Now, by cosine formula in ΔOAB, we have:

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2|OA|.|OB|} = \frac{r^2 + r^2 - r^2[2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2r.r}$$

$$\frac{1+1-[2-2(l_1l_2 + m_1m_2 + n_1n_2)]}{2} = \frac{2(l_1l_2 + m_1m_2 + n_1n_2)]}{2}$$

$$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|.$$

$$\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2).$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$[\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|]$$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$= l_1^2l_2^2 + l_1^2m_2^2 + l_1^2n_2^2 + m_1^2l_2^2 + m_1^2m_2^2 + m_1^2n_2^2 + n_1^2l_2^2 + n_1^2m_2^2 + n_1^2n_2^2$$

$$-(l_1^2l_2^2 + m_1^2m_2^2 + n_1^2n_2^2 + 2l_1l_2m_1m_2 + 2m_1m_2n_1n_2 + 2n_1n_2l_1l_2)$$

$$= (m_1^2n_2^2 + m_2^2n_1^2 - 2m_1m_2n_1n_2) + (n_1^2l_2^2 + n_2^2l_1^2 - 2n_1n_2l_1l_2) + (l_1^2m_2^2 + l_2^2m_1^2 - 2l_1l_2m_1m_2)$$

$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

But $0 \le \theta < \pi$ so that $\sin\theta > 0$.

Hence $$\sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}.$$

Tangent Form:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}.$$

Cor. 1. Condition of Perpendicular.

The two lines are orthogonal.

iff $$\theta = \frac{\pi}{2}$$

iff $$l_1l_2 + m_1\ m_2 + n_1n_2 = 0$$

Cor. 2. Condition of Parallelism.

The two lines are aligned.

Iff $\theta = 0°$

Iff $\sin\theta = 0$

$$\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} = 0$$

$$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 0.$$

$$m_1n_2 - m_2n_1 = 0, n_1l_2 - n_2l_1 = 0, l_1m_2 - l_2m_1 = 0.$$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{1}{1}$$

$$l_1 = l_2,\ m_1 = m_2 \text{ and } n_1 = n_2$$

(b) If $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ represent the direction ratios of lines L_1 and L_2 respectively, then the angle between them is given by:

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}.$$

Given that the direction ratios of L_1 and L_2 are $<a_1, b_1, c_1>$ and $<a_2, b_2, c_2>$ respectively.

The direction – cosines of L_1 and L_2 are:

$$< \pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} >$$

$$< \pm\frac{a_2}{\sqrt{a_2^2 + b_2{}^2 + c_2{}^2}}, \pm\frac{b_2}{\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}, \pm\frac{c_2}{\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}} >$$

[Ensure that all signs are either positive or negative]

The angle between the lines is determined by:

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}.$$

Note: If lines L_1 and L_2 are not perpendicular, then the acute angle between them is...
Given:

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

And obtuse angel between them is given by:

$$\cos\theta = -\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Sine Form:

Since $$\sin^2\theta = 1 - \cos^2\theta$$

$\therefore$ $$\sin^2\theta = 1 - \frac{(a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}.$$

$$= \frac{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}$$

$$= \frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{(a_1^2 + b^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}.$$

But $0 \le \theta < \pi$, so that $\sin\theta > 0$.

Hence, $$\sin\theta = \sqrt{\frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}.$$

Tangent Form:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\sqrt{\frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{a_1a_2 + b_1b_2 + c_1c_2}}$$

Cor.1. Condition of perpendicularity.

Lines L_1 and L_2 are perpendicular.

$$\theta = \frac{\pi}{2}$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Cor.2. Condition Of Parallelism.

Lines L_1 and L_2 are parallel.

$$\theta = 0°$$

$$\sin\theta = 0$$

$$\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2} = 0$$

$$(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$$

$$a_1b_2 - a_2b_1 = 0, b_1c_2 - b_2c_1 = 0, c_1a_2 - c_2a_1 = 0.$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$