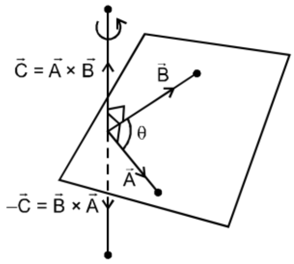
VECTOR ALGEBRA

VECTOR (OR CROSS) PRODUCT OF TWO VECTOR

VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT):

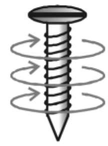
(A) If $\vec{a}$ & $\vec{b}$ are two vectors, and θ is the angle between them, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ form a right-handed screw system, is given by:



Sign convention

(A) Right handed screw system:

If $\vec{a}$, $\vec{b}$ and $\vec{n}$ form a right-handed system, it signifies that when we rotate vector $\vec{a}$ toward the direction of $\vec{b}$ by an angle θ, then $\hat{n}$ advances in the same direction as a right-handed screw would if turned in the same manner.



(B) Lagrange's Identity:

For any pair of vectors...

$$\vec{a} \,\&\, \vec{b}\ ;\ (\vec{a}\times\vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

(C) Formulation of vector product in terms of scalar product:

The cross product is a vector, such that...

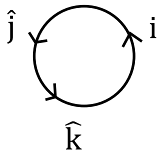
(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2}$

(ii) $\vec{c}\cdot\vec{a} = 0; \vec{c}\cdot\vec{b} = 0$

(iii) $\vec{a}, \vec{b}, \vec{c}$ form a right handed system

(D) (i) $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\ \&\ \vec{b}$ are parallel (collinear) $(\vec{a}\neq 0, \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is scalar.

(ii) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (Not commutative)



(iii) $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ Where m is scalar.

(iv) $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (Distributive over addition)

(v) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$

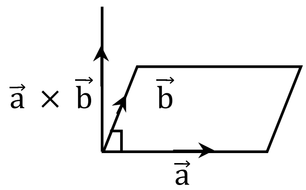
(vi) $\hat{i}\times\hat{j}=\hat{k}, \hat{j}\times\hat{k}=\hat{i}, \hat{k}\times\hat{i}=\hat{j}$

(E) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

& $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

(F) Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}\ \&\ \vec{b}$



(G) (i) Unit vector normal to the plane of... $\vec{a}\ \&\ \vec{b}$ is $n=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(ii) A vector with magnitude 'r' and perpendicular to the plane of

$$\vec{a}\ \&\ \vec{b} \text{ is } \pm\frac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

(iii) If θ is the angle between $\vec{a}\ \&\ \vec{b}$, then $\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(H) Vector area:

(i) If $\vec{a}, \vec{b}\ \&\ \vec{c}$ are the position vectors of three points A, B, and C respectively, then the vector area of...

$$\Delta ABC=\frac{1}{2}[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}].$$

Ex.1 Prove for any pair of vectors that... $(\bar{a}\cdot\bar{b})^2+(\bar{a}\times\bar{b})^2=|\bar{a}|^2|\bar{b}|^2$

Sol.
$$\left.\begin{aligned}(\vec{a}\cdot\vec{b})^2 &= |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta\\ (\vec{a}\times\vec{b})^2 &= |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta\end{aligned}\right\}$$

Where is the angle between $\vec{a}$ and $\vec{b}$.

Adding, $$\left(\vec{a}.\vec{b}\right)^2+\left(\vec{a}\times\vec{b}\right)^2=|\vec{a}|^2\,|\vec{b}|^2\left(\cos^2\theta+\sin^2\theta\right)$$

$$=|\vec{a}|^2\,|\vec{b}|^2$$

Ex.2 Find a unit vector that is perpendicular to the plane spanned by the vectors...

$\vec{a}=2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{b}=4\hat{i}+3\hat{j}-\hat{k}$.

Sol. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$\vec{c}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -6 & -3\\ 4 & 3 & -1\end{vmatrix}\div|\vec{a}\times\vec{b}|$$

$$=\frac{\hat{i}(15)-\hat{j}(10)+\hat{k}(30)}{|\vec{a}\times\vec{b}|}$$

$$=\frac{5(3\hat{i}-2\hat{j}+6\hat{k})}{5\cdot\sqrt{9+4+36}}$$

$$=\frac{(3\hat{i}-2\hat{j}+6\hat{k})}{7}$$

Alternate solution

Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$ be a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$\vec{c}\cdot\vec{a}=0 \text{ i.e. } 2x-6y-3z=0$$

$$\vec{c}\cdot\vec{b}=0 \text{ i.e. } 4x+3y-z=0$$

Solve for x, y and z; $$\frac{x}{15}=\frac{y}{-10}=\frac{z}{30}$$

$$\frac{x}{3}=\frac{y}{-2}=\frac{z}{6}$$

$$\frac{x}{3}=\frac{y}{-2}=\frac{z}{6}=\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{9+4+36}}=\frac{1}{7}$$

$$\vec{c}=\frac{(3\hat{i}-2\hat{j}+6\hat{k})}{7}$$

Ex.3 Show that if $\vec{a}$ be such that $|\vec{a}| \neq 0$ and the conditions $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ hold simultaneously, then $\vec{b} = \vec{c}$.

Sol. $$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0.$$

Either $|\vec{b} - \vec{c}| = 0$ or $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.

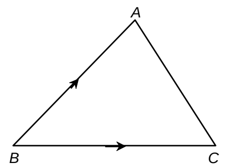
Also $$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = 0.$$

Either $|\vec{b} - \vec{c}| = 0$ or $\vec{a}$ is collinear to $\vec{b} - \vec{c}$.

If both are true the only possibility is $|\vec{b} - \vec{c}| = 0$ i.e. $\vec{b} = \vec{c}$.

Ex.4 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ and are the position vectors of three non-collinear points A, B, C respectively; indicate the nature of $\dfrac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{2}$

Sol.



$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$$
$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{a} - \vec{b}$$
$$\overrightarrow{BC} \times \overrightarrow{BA} = (\vec{c} - \vec{b}) \times (\vec{a} - \vec{b})$$
$$= \vec{c} \times \vec{a} - \vec{b} \times \vec{a} - \vec{c} \times \vec{b} \quad (\because \vec{b} \times \vec{b} = 0)$$
$$= \vec{c} \times \vec{a} + \vec{b} \times \vec{c} + \vec{a} \times \vec{b}$$
$$\frac{1}{2}(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = \frac{1}{2}\overrightarrow{BC} \times \overrightarrow{BA}$$
$$= \text{vector area of } \Delta ABC$$

Additionally, it is a vector that is perpendicular to the plane of the triangle.

Ex.5 If the vector $\vec{x}$ is perpendicular to the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$ and $\vec{x} \cdot (2\hat{i} - \hat{j} + \hat{k}) = -6$, find $\vec{x}$.

Sol. $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{vmatrix}$$
$$= \hat{i}(7) - \hat{j}(7) + \hat{k}(-7)$$
$$= 7(\hat{i} - \hat{j} - \hat{k})$$
$$= \lambda(\hat{i} - \hat{j} - \hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})$$
$$= \lambda(2 + 1 - 1) = -6$$
$$\lambda = -3$$

The required $\vec{x} = -3\hat{i} - 3\hat{j} + 3\hat{k}$