VECTOR ALGEBRA

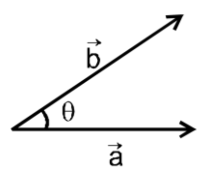
SCALAR (OR DOT) PRODUCT OF TWO VECTOR

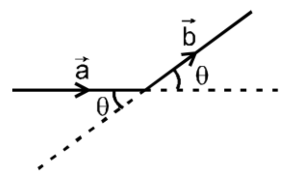
SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT)

Definition:

Consider two non-zero vectors and inclined at an angle θ. The scalar product of $\vec{a}$ with $\vec{b}$ is represented as $\vec{a}.\vec{b}$ and is defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta; 0 \le \theta \le \pi$$





Note:

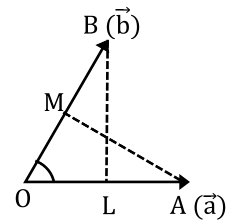
(A) If angle θ is acute, the dot product of vectors. $\vec{a}.\vec{b} > 0$; Conversely, if θ is obtuse, the $\vec{a}.\vec{b} < 0$.

(B) $\vec{a}.\vec{b} = 0 \; \vec{a} \perp \vec{b} \; (\vec{a} \neq 0, \; \vec{b} \neq 0)$

(C) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}||\vec{b}|$

(D) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}||\vec{b}|$

Geometrical interpretation of scalar product



Let vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be representations of vectors $\vec{A}$ and $\vec{B}$, respectively.

Let θ represent the angle between vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

Draw BL $\perp$ OA and AM$\perp$ OB.

From ΔOBL and ΔOAM,

We have OL $=$ OB $\cos\theta$ and OM $=$ OA $\cos\theta$.

In this context, OL is referred to as the projection of vector $\vec{b}$ on $\vec{a}$, and OM is the projection of vector $\vec{a}$ on $\vec{b}$.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|(|\vec{b}|\cos\theta)$$

$$= |\vec{a}|(OB\cos\theta) = |\vec{a}|(OL)$$

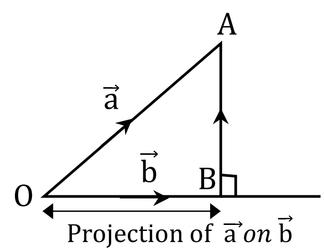
$$= (\text{ Magnitude of }\vec{a})(\text{ Projection of }\vec{b}\text{ on }\vec{a}) \quad \ldots\ldots\ldots \text{(i)}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}|\left(\left|\frac{1}{a}\right|\cos\theta\right)$$

$$= |\vec{b}|(OA\cos\theta) = |\vec{b}|(OM)$$

$$= (\text{ magnitude of }\vec{b})\text{ Projection of }\vec{a}\text{ on }\vec{b}) \qquad \ldots\ldots\text{(ii)}$$

Therefore, from a geometric perspective, the scalar product of two vectors is the result of multiplying the magnitude of one vector by the projection of the other vector in its direction.



(i) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad (0 \le \theta \le \pi)$

Note that If θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

(2) (i) $\vec{a}.\vec{a} = |\vec{a}| = (\vec{a})^2$

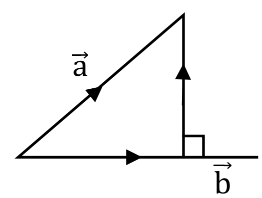
(ii) $\vec{a}\times\vec{b} = \vec{b}\times\vec{a}$ (Commutative)

(3) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive)

(4) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\perp\vec{b};(\vec{a},\vec{b}\neq 0)$

(5) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 1$

$\hat{i}\times\hat{j} = \hat{j}\times\hat{k} = \hat{k}\times\hat{i} = 0$



(6) $(m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b}) = m(\vec{a}\cdot\vec{b})$ where m is a scalar.

(7) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ (Provided $|\vec{b}|\neq 0$)

(8) The vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to

$\vec{b} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ [by triangle law of vector Addition]

(9) The angle ∅ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\, 0 \le \phi \le \pi$

(10) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1\ b_1 + a_2\ b_2 + a_3\ b_3$

$|\vec{a}| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}}, |\vec{b}| = \sqrt{b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2}}$

(11) Maximum value of $\vec{a}\cdot\vec{b} = \vec{a}||\vec{b}|$

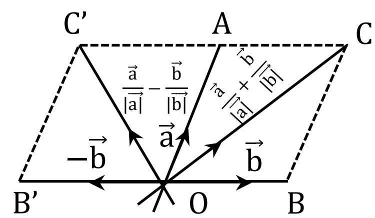
(12) Minimum value of $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

(13) Any vector $\vec{a}$ can be written as $\vec{a} = (a\cdot\hat{i})\hat{i} + (a\cdot\hat{j})\hat{j} + (\vec{a}\cdot\hat{k})\hat{k}$

(14) $|\vec{a}\pm\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$, where θ is the angle between the vectors.

VECTOR EQUATION OF ANGLE BISECTOR

A vector along the bisector of the angle between two vectors $\vec{a}$ & $\vec{b}$ is represented by $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$.



Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$

Bisector of the exterior angle of $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$

Ex.1 Determine the value of p that satisfies the vectors...

$$\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k} \text{ and } \vec{b} = \hat{i} + p\hat{j} + 3\hat{k} \text{ are}$$

(i) Perpendicular (ii) parallel

Sol. (i)

$$\vec{a} \perp \vec{b}$$

$$\vec{a}\cdot\vec{b} = 0$$

$$(3\hat{i} + 2\hat{j} + 9\hat{k})\cdot(\hat{i} + p\hat{j} + 3\hat{k}) = 0$$

$$3 + 2p + 27 = 0$$

$$P = -15$$

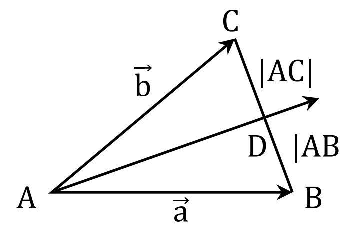
(ii) Vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3}$$

$$3 = \frac{2}{p}$$

$$p = \frac{2}{3}$$

Ex.2 The vector, $\vec{c}$, oriented along the internal bisector of the angle between the vector $7\hat{i} - 4\hat{j} - 4\hat{k}$ And $-2\hat{i} - \hat{j} + 2\hat{k}$ with $|\vec{c}| = 5\sqrt{5}$ is.



Sol. Let $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$

And $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$

Internal bisector divides the BC in the ratio of $|\overrightarrow{AB}| : |\overrightarrow{AC}|$

$$|\overrightarrow{AB}| = 9, |\overrightarrow{AC}| = 3$$

$$|\overline{AD}| = \left(\frac{9(-2\hat{i} - \hat{j} + 2\hat{k}) + 3(7\hat{i} - 4\hat{j} - 4\hat{k})}{9+3} \right) = \frac{\hat{i} - 7\hat{j} + 2\hat{k}}{4}$$

$$\vec{c} = \pm \left(\frac{\overline{AD}}{|\overline{AD}|} \right) 5\sqrt{6} = \pm \frac{5}{3}(\hat{i} - 7\hat{j} + 2k)$$

Ex.3 If $a = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ then find

(i) Component of $\vec{b}$ along $\vec{a}$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$

Sol. (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \vec{a}$

$$\vec{a} \cdot \vec{b} = 2 - 1 + 3 = 4$$

$$|\vec{a}|^2 = 3$$

$$\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2} \right) \vec{a} = \frac{4}{3} \vec{a} = \frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$$

(ii) The projection of vector $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is

$$\vec{b} - \left(\frac{a \cdot \hat{b}}{|a|^2} \right) \vec{a} \cdot = \frac{1}{3}(2\hat{i} - 7\hat{j} + 5\hat{k})$$

Ex.4 If the magnitudes of vectors $\vec{a}, \vec{b}, \vec{c}$ are 3, 4 and 5 respectively, and $\vec{a}$ is related to $\vec{b}+\vec{c}$, and $\vec{b}$ is related to and $\vec{C}+\vec{a}$, $\vec{c}$ and $\vec{a}+\vec{b}$ are perpendicular to each other, then modulus of $\vec{a}+\vec{b}+\ \vec{c}$ is.

Sol.

$$\vec{a} \perp (\vec{b}+\vec{c})$$
$$\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$$
$$\vec{b} \perp (\vec{c}+\vec{a})$$
$$\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}=0$$
$$\vec{c} \perp (\vec{a}+\vec{b})$$
$$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0$$
$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0$$
$$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$
$$= 9 + 16 + 25 = 50$$
$$|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$$

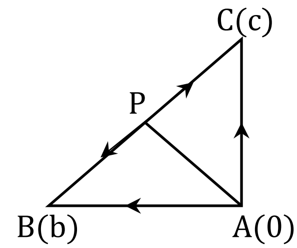
Ex.5 If $\vec{a}+\vec{b}+\vec{c}=\vec{0}, |\vec{a}|=3, |\vec{b}|=5$ and $|\vec{c}|=7$, Solve the angle between $\vec{a}$, and $\vec{b}$.

Sol. We have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\vec{a}+\vec{b}=-\vec{c}$$
$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$$
$$|\vec{a}+\vec{b}|^2=|\vec{c}|^2$$
$$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2$$
$$|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta=|\vec{c}|^2$$
$$9 + 25 + 2(3)\ (5)\cos\theta = 49$$
$$\cos\theta=\frac{1}{2}$$
$$\theta=\frac{\pi}{3}$$

Ex.6 Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

Sol. Due to the isosceles nature of the triangle, we obtain...



$$AB = AC \qquad \text{......(i)}$$

Now. $\overrightarrow{AP} = \dfrac{\vec{b}+\vec{c}}{2}$ where P is mid-point of BC.

Also $$\overrightarrow{BC} = \vec{c} - \vec{b}$$

$$\overrightarrow{AP}\cdot\overrightarrow{BC} = \frac{\vec{b}+\vec{c}}{2}\cdot(\vec{c}-\vec{b}) = \frac{1}{2}\left(c^2 - b^2\right)$$

$$= \frac{1}{2}\left(AC^2 - AB^2\right) = 0 \qquad \{\text{by (i)}\}$$

Median AP is perpendicular to base BC.