VECTOR ALGEBRA

MULTIPLICATION OF A VECTOR BY A SCALAR

(C) Multiplication of vector by scalars

If $\vec{a}$ is a vector and m is a scalar, then the product $m(\vec{a})$ results in a vector parallel to , with a magnitude equal to |m| times that of .

This operation is referred to as scalar multiplication. If $\vec{a}$ and $\vec{b}$ are vectors, and m and n are scalars, then the result is expressed as:

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$

(ii) $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(iv) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

Note: In general, for any non-zero vectors, $\vec{a}, \vec{b} \,\&\, \vec{c}$ one may note that although $\vec{a}+\vec{b}+\vec{c}=0$ it is important to observe that while, it does not necessarily form the three sides of a triangle.

Ex.1 ABCD is a parallelogram with intersecting diagonals at P. If O is a constant point, then the expression is $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ equivalent to.

Sol. Since, P bisects both the diagonal AC and BD,

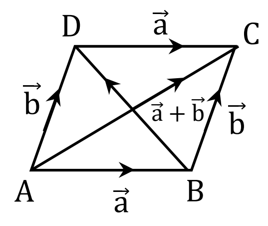
so $$\overline{OA}+\overline{OC}=2\overline{OP}$$

And $$\overline{OB}+\overline{OD}=2\overrightarrow{OP}$$

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$$

Ex.2 Given $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ that represent two adjacent sides of a parallelogram, determine unit vectors that are parallel to the diagonals of the parallelogram.

Sol. Let ABCD be a parallelogram such that



$$\overrightarrow{AB}=\vec{a} \text{ and } \overrightarrow{BC}=\vec{b}.$$

Then $$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\overrightarrow{AC}=\vec{a}+\vec{b}=3\hat{i}+6\hat{j}-2\hat{k}$$

$$|\overrightarrow{AC}|=\sqrt{9+36+4}=7$$

$$\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$$

$$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}$$

$$= \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$$

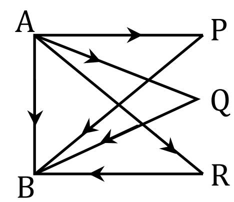
$$|\overline{BD}| = \sqrt{1+4+64} = \sqrt{69}$$

Unit vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$

Unit vector along $\overrightarrow{BD} = \dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \dfrac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} - 8\hat{k})$

Ex.3 A, B, P, Q, R are five points in a plane. If force $\overrightarrow{AP}, \overrightarrow{AQ}, \overrightarrow{AR}$ is applied to point A and force $\overline{PB}, \overline{QB}, \overline{RB}$ is applied to point B, then the resultant force is represented by.

Sol. From figure



$$\overline{AP} + \overline{PB} = \overline{AB}$$

$$\overline{AQ} + \overrightarrow{QB} = \overrightarrow{AB}$$

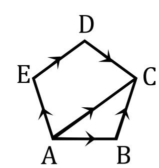
$$\overline{AR} + \overrightarrow{RB} = \overrightarrow{AB}$$

So, $$(\overline{AP} + \overline{AQ} + \overline{AR}) + (\overline{PB} + \overrightarrow{QB} + \overline{RB}) = 3\overrightarrow{AB}$$

So, required resultant $= 3\overrightarrow{AB}$

Ex.4 ABCDE is a pentagon. Demonstrate that the resultant of the forces $\overrightarrow{AB}, \overrightarrow{AE}, \overrightarrow{BC}, \overrightarrow{DC}, \overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$. is.

Sol. Let $\vec{R}$ be the resultant force



$$\vec{R} = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$$

$$\vec{R} = (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$$

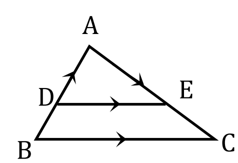
$$\vec{R} = \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$$

$$R = 3\overrightarrow{AC}$$ **Hence proved**

Ex.5 Demonstrate that the line connecting the midpoints of two sides of a triangle is parallel to the third side and is half of its length.

Sol. Consider the midpoints of side AB and AC of triangle ABC as D and E, respectively.

Now in ΔABC, by triangle law of addition



$$\overrightarrow{BA} = 2\overrightarrow{DA}$$

$$\overrightarrow{AC} = 2\overrightarrow{AE}$$

Now, in ΔABC,, applying the triangle law of addition

$$\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$$

$$2\overrightarrow{DA} + 2\overrightarrow{AE} = \overrightarrow{BC}$$

$$\overrightarrow{DA}+\overrightarrow{AE}=\frac{1}{2}\overrightarrow{BC}$$

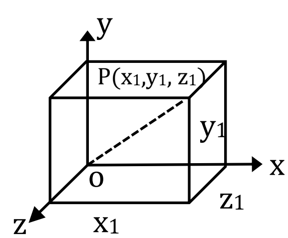
$$\overrightarrow{DE}=\frac{1}{2}\overrightarrow{BC}$$

Therefore, the line DE is parallel to the third side BC of the triangle and is half of its length.

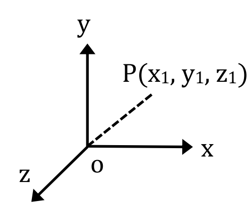
COORDINATE OF A POINT IN SPACE

In space, an infinite number of points exist. To uniquely identify each point, three mutually perpendicular coordinate axes, namely OX, OY, and OZ, are employed. These three axes, represented by lines, intersect at the origin. The plane formed by the x and y axes is termed the x-y plane, the y and z axes create the y-z plane, and the z and x axes give rise to the z-x plane.

Take any point P in space. Project a perpendicular from this point to the x-y plane. The algebraic length of this perpendicular is regarded as the z-coordinate. From the base of this perpendicular, project additional perpendiculars to the x and y axes. The algebraic lengths of these perpendiculars are considered as the y and x coordinates, respectively.



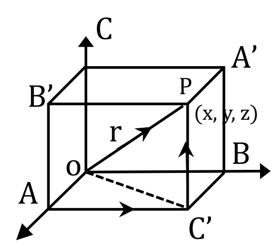
VECTOR REPRESENTATION OF A POINT IN SPACE



If the coordinates of a point P in space are (x, y, z), then the position vector of point P relative to the same origin is $x\hat{i}+y\hat{j}+z\hat{k}$ represented by.

REPRESENTATION OF A VECTOR IN SPACE

Orthonormal triad of unit vectors



Consider a point P(x, y, z) in space, where OX, OY, and OZ serve as the coordinate axes.

Then $OA = x, OB = y$ and $OC = z$

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively,

Then $\overline{OA}=x\hat{i},\overline{OB}=y\hat{j},\overline{OC}=z\hat{k}$

$$\overrightarrow{OP}=\overline{OC'}+\overline{C'P}=\overline{OB}+\overline{OA}+\overline{OC}$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \left[\because \overline{C'P} = \overline{OC}\right]$$

$$\overrightarrow{OP} = \vec{r}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = OP = \sqrt{x^2 + y^2 + z^2}$$

Distance of a point P from coordinate axes

Let PA, PB, and PC represent the distances of the point P(x, y, z) from the coordinate axes OX, OY, and OZ, respectively.

$$PA = \sqrt{y^2 + z^2}\ ,\ PB = \sqrt{z^2 + x^2}\ ,\ PC = \sqrt{x^2 + y^2}$$

Distance formula

The distance between two points A $A(\vec{a})$ and $B(\vec{b})$ is $AB = |\vec{a} - \vec{b}|$.

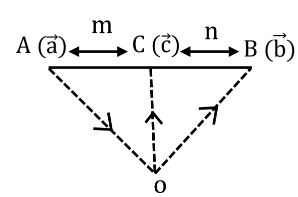
The distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is expressed as:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

SECTION FORMULA

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A and B, then the position vector $(\vec{r})$ of a point C, which divides AB in the ratio m : n, is given by:

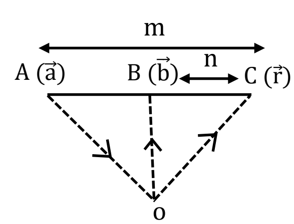
(A) Internal Division:



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

Position vector of midpoint of $AB = \dfrac{\vec{a} + \vec{b}}{2}$

(B) External division:



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Ex.6 Demonstrate that the points (0, 7, 10), (–1, 6, 6), and (–4, 9, 6) constitute a right-angled isosceles triangle.

Sol. Let A = (0, 7, 10), B = (–1, 6, 6), C= (–4, 9, 6)

$$AB^2 = (0 + 1)^2 + (7 - 6)^2 + (10 - 6)^2 = 18$$

$$AB = 3\sqrt{2}$$

Similarly $BC = 3\sqrt{2}$

And $AC = 6$

Clearly $AB^2 + BC^2 = AC^2$

$\therefore \quad \angle ABC = 90°$

Also $\quad AB = BC$

Hence ΔABC is right angles isosceles.

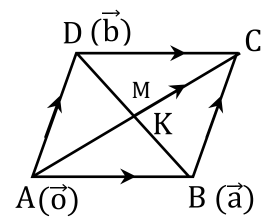
Ex.7 If the midpoints of sides BC, CA, and AB of triangle ABC are denoted as D, E, and F, respectively, then the position vector of the centroid of triangle DEF, given that the position vectors of A, B, and C are i + j, j + k, and k + i, respectively, is:

Sol. The position vector of points D, E, F are respectively $\frac{i+j}{2}+k, i+\frac{k+j}{2}$ and $\frac{i+k}{2}+j$

So, position vector of centroid of

$$\Delta DEF = \left[\frac{i+j}{2}+k+i+\frac{k+j}{2}+\frac{i+k}{2}+j\right] = \frac{2}{3}[i+j+k]$$

Ex.8 If ABCD is a parallelogram and E is the midpoint of AB, demonstrate using vector methods that DE trisects AC and is also trisected by AC.



Sol. Let $\quad \overrightarrow{AB} = \vec{a}$ and $AD = \vec{b}$

Then $\quad \overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$

And $\quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

Also let K be a point on AC, such that AK : AC = 1 : 3

$$AK = \tfrac{1}{3} AC$$

$$\overrightarrow{AK} = \frac{1}{3}(\vec{a}+\vec{b}) \qquad \text{.......(i)}$$

Again E being the midpoint of AB,

We have, $\quad \overrightarrow{AE} = \frac{1}{2}\vec{a}$

Let M be the point on DE such that DM: ME = 2: 1

$$\overline{AM} = \frac{\overrightarrow{AD}+2\overrightarrow{AE}}{1+2} = \frac{\vec{b}+\vec{a}}{3} \qquad \text{.......(ii)}$$

From eq. (i) and (ii) we found that

$$\overline{AK} = \frac{1}{3}(\vec{a}+\vec{b}) = \overline{AM},$$

And so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

Ex.9 Demonstrate, using the distance formula, that the points (4, 5, –5), (0, –11, 3), and (2, –3, –1) lie on the same straight line (are collinear).

Sol. Let A = (4, 5, –5) , B = (0, –11, 3), C = (2, –3, –1)

$$AB = \sqrt{(4-0)^2 + (5+11)^2 + (-5-3)^2} = \sqrt{336} = \sqrt{4 \times 84} = 2\sqrt{84}$$

$$BC = \sqrt{(0-2)^2 + (-11+3)^2 + (3+1)^2} = \sqrt{84}$$

$$AC = \sqrt{(4-2)^2 + (5+3)^2 + (-5+1)^2} = \sqrt{84}$$

$$BC + AC = AB$$

Hence, points A, B, C are collinear and C lies between A and B.