SEQUENCES AND SERIES

SUM TO N TERMS OF SPECIAL SERIES (For competitive exam)

Sum of n Terms of Special Series:

A series is the aggregate of all the numbers within a given sequence, with sequences existing in both finite and infinite forms. Similarly, a series itself can be either finite or infinite.

For example,

Take a sequence such as 1, 3, 5, 7, ... The corresponding series formed by adding these terms together is $1 + 3 + 5 + 7 + \ldots$ A series that possesses unique characteristics or significance is referred to as a.

Special series.

The following are the three types of special series.

1. The sum of the first n natural numbers expressed as $1 + 2 + 3 + \ldots + n$.
2. The sum of the squares of the first n natural numbers represented as $1^2 + 2^2 + 3^2 + \ldots + n^2$.
3. The sum of the cubes of the first n natural numbers expressed as $1^3 + 2^3 + 3^3 + \ldots + n^3$.

In this article, we will explore how to derive the formulas for each of these series.

Special Series 1: Sum of first n natural numbers

Below is the outcome of this series:

$$1 + 2 + 3 + 4 + \ldots. + n = \frac{n(n+1)}{2}$$

Proof:

Let $S_n = 1 + 2 + 3 + 4 + \ldots + n$

It is evident that this constitutes an Arithmetic Progression, where the initial term (a) is equal to 1.

The common difference (d) is 1, and the sequence comprises a total of n terms.

So, Sum of n terms $= \frac{n}{2}(2 \times a + (n - 1) \times d)$

Substituting the values into this series will yield:

$$S_n = \frac{n}{2}(2 \times 1 + (n-1) \times 1)$$

$$S_n = \frac{n}{2}(2 + n - 1)$$

$$S_n = \frac{n(n+1)}{2}$$

Hence Proved.

Ex.1 Determine the sum of the series: 3 + 4 + 5 + ... + 25.

Sol. Let $S_n = 3 + 4 + 5 — + 25$

Now, we can express it in the following manner:

$$S_n + 1 + 2 = 1 + 2 + 3 + 4 —- + 25$$

Clearly, it is now the sum of the first 25 natural numbers, and it can be represented as follows:

$$S_n + 1 + 2 = \frac{25(25+1)}{2}$$

$$S_n = 325 - 1 - 2$$

$$S_n = 322$$

Special Series 2: **Sum of squares of the first n natural numbers**

Below is the outcome of this series:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof:

Let $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$ .... (1)

We know that, $k^3 - (k-1)^3 = 3k^2 - 3k + 1$ .....(2)

We know that, $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

So, $k^3 - (k-1)^3$

$k^3 - k^3 + 1 + 3k^2 - 3k$

$3k^2 - 3k + 1$

Putting k = 1, 2…, n successively in eq 2, we obtain

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$

$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$

$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$

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$$n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$$

Summing both sides of the equations above, we obtain:

$$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + \ldots + n^2)\; - 3\,(1 + 2 + 3 + \ldots + n) + n$$

This can be expressed as:

$$n^3 = 3\sum(k^2) - 3\sum(k) + n, \text{ where } 1 \le k \le n \qquad .....(3)$$

We know that,

$$\sum(k)\;\; (\text{where } 1 \le k \le n\,) = 1 + 2 + 3 + 4 — n = \frac{n(n+1)}{2} \;.....(4)$$

And eq 1 can also be written like this

$$S_n = \sum(k^2), \text{ where } 1 \le k \le n$$

Now, putting these values in eq 3

$$n^3 = 3S_n - \frac{3(n)(n+1)}{2} + n$$

$$n^3 + \frac{3(n)(n+1)}{2} - n = 3S_n$$

$$\frac{2n^3+3n^2+3n-2n}{2} = 3S_n$$

$$\frac{2n^3+3n^2+n}{6} = S_n$$

$$\frac{n\left(2n^2+3n+1\right)}{6} = S_n$$

$$\frac{n\left(2n^2+n+2n+1\right)}{6} = S_n$$

$$\frac{n\left(n(2n+1)+1(2n+1)\right)}{6} = S_n$$

$$\frac{n(n+1)(2n+1)}{6} = S_n$$

$$S_n = \frac{n(n+1)(2n+1)}{6}$$ **Hence proved.**

Ex.2 Determine the sum of the n terms in the series, where the nth term is given by $n^2 + n + 1$.

Sol. Given that , $a_n = n^2 + n + 1$

Thus, the sum to n terms is given by

$$S_n = \sum ak \qquad (\text{where } 1 \le k \le n\,)$$

$$= \sum k^2 + \sum k + \sum 1 \qquad (\text{where } 1 \le k \le n)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)+3n(n+1)+6n}{6}$$

$$= \frac{(n^2+n)(2n+1)+3n^2+3n+6n}{6}$$

$$= \frac{2n^3+2n^2+n^2+n+3n^2+9n}{6}$$

$$= \frac{2n^3+6n^2+10n}{6}$$

Ex.3 Calculate the sum of the series up to the nth term.

$$1 + 1 + 2 + 1 + 2 + 3 + 1 + 2 + 3 + 4 + \text{——}?$$

Sol. Upon careful observation of the series, we can express it in the following manner:

$$S_n = (1) + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \text{——}$$

We can state that the task is to calculate the sum of the first n natural numbers.

So we can write $S_n = \Sigma\left(\frac{\mathbf{i}(\mathbf{i}+1)}{2}\right)$,

Where $1 \le i \le n$

$$= (\tfrac{1}{2})\sum\big(i(i+1)\big)$$

$$\left(\frac{1}{2}\right)\sum(i^2 + i)$$

$$\left(\frac{1}{2}\right)(\Sigma\mathbf{i}^2 + \Sigma\mathbf{i})$$

We know $\Sigma i^2 = \frac{n(n+1)(2n+1)}{6}$

and $\Sigma i = \frac{n(n+1)}{2}$.

Substituting the value,

we get, $\text{Sum} = \left(\frac{1}{2}\right)\left(\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)\right)$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{6} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

Special Series 3: Sum of cubes of the first n natural numbers

The outcome of this series is provided below:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Proof:

Let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3 \quad \ldots (1)$

We know that, $(k + 1)^4 - (k)^4 = 4k^3 + 6k^2 + 4k + 1 \quad \ldots.. (2)$

We know that, $(a + b)^4 = (a^2 + b^2 + 2ab)^2$

$$= a^4 + b^4 + 6a^2b^2 + 4a^3b + 4ab^3$$

So, $(k + 1)^4 - (k)^4$

$$= k^4 + 1 + 6k^2 + 4k^3 + 4k - k^4$$

$$= 4k^3 + 6k^2 + 4k + 1$$

Putting k = 1, 2…, n successively in eq 2 ,

we obtain $(1 + 1)^4 - 1^4 = 4(1)^3 + 6(1)^2 + 4(1) + 1$

$$(2 + 1)^4 - 2^4 = 4(2)^3 + 6(2)^2 + 4(2) + 1$$

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$$(n + 1)^4 - (n)^4 = 4(n)^3 + 6n^2 + 4n + 1$$

Adding both sides of all the above equations,

we get $(n + 1)^4 - 1^4$

$$= 4(1^3 + 2^3 + 3^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 4(1 + 2 + 3 + \ldots + n) + n$$

We can write this like:

$$(n + 1)^4 - 1^4 = 4 \sum (k^3) + 6\sum(k^2) + 4\sum(k) + n$$

where $1 \le k \le n$ ….(3)

We know that ,

$\sum(k)$ (where $1 \le k \le n$) $= 1 + 2 + 3 + 4 — n = \frac{n(n+1)}{2}$ ….(4)

$\sum(k^2)$ (where $1 \le k \le n$) $= 1^2 + 2^2 + 3^2 + 4^2 — n^2 = \frac{n(n+1)(2n+1)}{6}$ ….(5)

and eq1 can also be written like this

$$S_n = \sum(k^3) \text{ , where } 1 \le k \le n$$

Now, putting these values in eq 3

$$(n + 1)^4 - 1^4 = 4S_n + \frac{6(n)(n+1)(2n+1)}{6} + \frac{4(n)(n+1)}{2} + n$$

$$n^4 + 6n^2 + 4n^3 + 4n - (n)(2n^2 + 3n + 1) - 2(n)(n + 1) - n = 4S_n$$

$$n^4 + 6n^2 + 4n^3 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 2n - n = 4S_n$$

$$n^4 + n^2 + 2n^3 = 4S_n$$

$$n^2 (n^2 + 1 + 2n) = 4S_n$$

$$n^2 (n + 1)^2 = 4S_n$$

$S_n = \left(\frac{n(n+1)}{2}\right)^2$ **Hence proved.**

Ex.4 Determine the value of the given fraction.

$$\frac{\left(1^3 + 2^3 + 3^3 + \ldots. + 9^3\right)}{1 + 2 + 3 + \ldots. + 9} = ?$$

Sol. Sum of first n natural number : $\frac{n(n+1)}{2}$

Sum of cube of first n natural number : $\left(\frac{n(n+1)}{2}\right)^2$

So, $$\frac{1^3+2^3+3^3--n^3}{1+2+3--+n} = \left(\frac{\frac{n(n+1)}{2)^2}}{\frac{n(n+1)}{2}}\right) = \frac{n(n+1)}{2}$$

Now, as we can see that value of n is 9 in the question,

$$= \frac{9(9+1)}{2} = 9 \times 5 = 45$$