# SEQUENCES AND SERIES

## RELATION BETWEEN AM & GM

## Relations between A.M. and G.M.

For two positive real numbers, a and b

$$A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A > G \quad \text{if} \quad a \neq b \qquad \text{...(i)}$$

$$A = G \quad \text{if} \quad a = b \qquad \text{...(ii)}$$

So, combining (i) & (ii),

We have $A \geq G$, and equality holds when $a = b$.

If $a_1, a_2, a_3 ... a_n$ are n positive numbers,

Then, Above discussion leads to the result that,

$$\frac{a_1+a_2+a_3+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$

**Ex.1** There are n arithmetic means between 1 and 31, and the 7th mean to the $n - 1^{th}$ mean forms a ratio of º 5 : 9. Find n ?

**Sol.** Let d and $A_j$ denote the common difference and $j^{th}$ Arithmetic mean respectively ;

then, $d = \frac{31-1}{n+1} = \frac{30}{n+1}$

$\Rightarrow \quad A_7 = 1 + 7\frac{30}{n+1} = 1 + \frac{210}{n+1}$

$\Rightarrow \quad A_{n-1} = 1 + (n-1)\frac{30}{n+1}$

$\Rightarrow \quad \frac{A_7}{A_{n-1}} = \frac{5}{9}$

$\Rightarrow \quad 9 + \frac{1890}{n+1} = 5 + \frac{150(n-1)}{n+1}$

$\Rightarrow \quad 146\,n = 2044$

$\Rightarrow \quad n = 14.$

**Ex2.** If one arithmetic mean (A.M.), A, and two geometric means (G.M.s) p and q are inserted between any two given numbers, then it can be shown that $p^3 + q^3 = 2Apq$.

**Sol.** Let the two given numbers be a and b ;

then, $2A = a + b$ ...(1)

a, p, q, b are in G.P.

$\Rightarrow$ $p^2 = aq$ and $q^2 = bp$

$\Rightarrow$ $p^3 = apq$ and $q^3 = bpq$

$\therefore$ $p^3 + q^3 = (a + b)\,pq = 2A\,pq$

## Special Series

**Sigma (S) notation** : S indicates sum i.e., $\sum_{i=1}^{n} i = \sum n = 1 + 2 + 3 + \cdots \ldots + n$

(i) $\sum_{l=1}^{n} \frac{i+1}{i+2} = \frac{1+1}{1+2} + \frac{2+1}{2+2} + \frac{3+1}{3+2} + \cdots . + \frac{n+1}{n+2}$

(ii) $\sum_{1=1}^{m} a = a + a + \cdots \ldots + a$ m times $= am$ where a is constant

(iii) $\sum_{l=1}^{m} ai = a\sum_{l=1}^{m} i = a(1 + 2 + \cdots . + m)$

(iv) $\sum_{l=1}^{m} (i^3 - 2i^2 + i) = \sum_{l=1}^{m} i^3 - 2\sum_{l=1}^{m} i^2 + \sum_{l=1}^{m} i$

## Important Results

(i) Sum of the first n natural numbers.

$$\Sigma n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(ii) Sum of the squares of the first n natural numbers.

$$Sn^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of the cubes of the first n natural numbers.

$$Sn^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$$

(iv) Sum of the first n terms of a sequence $T_n = an^3 + bn^2 + cn + d$

$$S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn$$

**Ex.3** Calculate the sum of the series 3.5 + 6.8 + 9.11 + ... up to n terms.

**Sol.** $n^{th}$ term of 3, 6, 9, .... is 3n

$n^{th}$ term of 5, 8, 11, .... is (3n + 2)

$\therefore$ $T_n = 3n\,(3n + 2) = 9n^2 + 6n$

$\therefore$ $S_n = 9Sn^2 + 6Sn$

$$= \frac{9n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2}$$

$$= \frac{3}{2} n(n+1)[2n+1+2]$$

$$= \frac{3n(n+1)(2n+3)}{2}$$

## ARITHMETICO-GEOMETRIC SERIES (A.G. S.) (For competitive exam)

$n^{th}$ term of A.G.. S. = ($n^{th}$ term of an A.P.) × ($n^{th}$ term of a G.P.)

If a, (a + d), (a + 2d) + .... be an A.P. & b, br, $br^2$ + ..... be a G.P.

Then ab + (a + d) br + (a + 2d)$br^2$ + .... is the corresponding A.G.S.

$$T_n \text{ of A.G.S.} = (T_n \text{ of A.P.}) \times (T_n \text{ of G.P.})$$