SEQUENCES AND SERIES

GEOMETRIC PROGRESSION

GEOMETRIC PROGRESSION:

A geometric sequence, or geometric progression, is a sequence of numbers in which each successive number is the product of the previous number and a constant r.

$a_n = ra_{n-1}$ **Geometric Sequence**

And because $\frac{a_n}{a_{n-1}} = \mathbf{r}$, the constant factor r is called the common ratio.

For instance, the following is an example of a geometric sequence: 9, 27, 81, 243, 729...

Here, where a_1 is 9, and the ratio between any two successive terms is 3.

Construct the general term $a_n=3a_{n-1}$

$$a_1 = 9$$
$$a_2 = 3a_1 = 3(9) = 27$$
$$a_3 = 3a_2 = 3(27) = 81$$
$$a_4 = 3a_3 = 3(81) = 243$$
$$a_5 = 3a_4 = 3(243) = 729$$

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In general, if we have the first term a_1 and the common ratio r of a geometric sequence, we can express it as follows:

$$a_2 = ra_1$$
$$a_3 = ra_2 = r(a_1r) = a_1r^2$$
$$a_4 = ra_3 = r(a_1r^2) = a_1r^3$$
$$a_5 = ra_3 = r(a_1r^3) = a_1r^4$$

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From this, we observe that any geometric sequence can be represented in terms of its initial element, common ratio, and index as follows:

$a_n = a_1 r^{n-1}$ **Geometric Sequence**

In fact, any general term that is exponential in n is a geometric sequence.

Ex.1 Determine the general term equation for the provided geometric sequence and utilize it to compute the 10th term: 3, 6, 12, 24, 48…

Sol. Begin by finding the common ratio, $r = \frac{6}{3} = 2$.

Recognize that the ratio between consecutive terms is 2. This confirms that the sequence is a geometric progression with a first term (a_1) of 3 and a common ratio (r) of 2.

$$a_n = a_1 r^{n-1}$$

$$3(2)^{n-1}$$

Hence, the general term is given by an $= 3(2)^{n-1}$, and the 10th term can be determined as follows:

$$a_{10} = 3(2)^{10-1} = 3(2)^9 = 1{,}536$$

Answer: $a_n=3(2)^{n-1}$; $a_{10}=1{,}536$

The terms located between given terms of a geometric sequence are referred to as geometric means.

Ex.2 Find all terms between $a_1=-5$ and $a_4=-135$of a geometric sequence. In other words, find all geometric means between the 1st and 4th terms.

Sol. Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$a_n = a_1 r^{n-1} \text{ use } n = 4$$

$$a_4 = a_1 r^{4-1}$$

$$a_4 = a_1 r^3$$

Substitute $a_1=-5$and $a_4=-135$ into the above equation and then solve for r.

$$-135 = -5\, r^3$$

$$27 = r^3$$

$$3 = r$$

Next, utilize the first term ($a_1 = -5$) and the common ratio ($r = 3$) to establish an equation for the n^{th} term in the sequence.

$$a_n = a_1 r^{n-1}$$

$$a_n = -5(3)^{n-1}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

Now, we can employ the equation ($a_n = -5\ (3)^{n-1}$), where n is a positive integer, to ascertain the values of the omitted terms.

$$a_1 = -5(3)^{1-1} = -5 \cdot 3^0 = -5$$
$$a_2 = -5(3)^{2-1} = -5 \cdot 3^1 = -15$$
$$a_3 = -5(3)^{3-1} = -5 \cdot 3^2 = -45$$
$$a_4 = -5(3)^{4-1} = -5 \cdot 3^3 = -135$$

Answer: -15 , -45

The first term of a geometric sequence may not be given.

Ex.3 Find the general term of a geometric sequence where $a_2 = -2$ and $a_5 = \frac{2}{125}$

Sol. To establish a formula for the general term, we need values for a_1 and r. A system of nonlinear equations can be constructed using the provided information and the expression $a_n = a_1\, r^{n-1}$.

$$\begin{cases} a_2 = a_1 r^{2-1} \\ a_5 = a_1 r^{5-1} \end{cases} \Rightarrow \begin{cases} -2 = a_1 r & \text{use } a_2 = -2 \\ \dfrac{2}{125} = a_1 r^4 & \text{use } a_5 = \dfrac{2}{125} \end{cases}$$

Solve for a_1 in the first equation,

$$\begin{cases} -2 = a_1 r & \Rightarrow \dfrac{-2}{r} = a_1 \\ \dfrac{2}{125} = a_1 r^4 \end{cases}$$

Substitute $a_1 = -\frac{2}{r}$ into the second equation and solve for r.

$$\frac{2}{125} = a_1 r^4$$
$$\frac{2}{125} = \left(\frac{-2}{r}\right) r^4$$
$$\frac{2}{125} = -2r^3$$
$$-\frac{1}{125} = r^3$$
$$-\frac{1}{5} = r$$

Back substitute to find a_1:

$$a_1 = \frac{-2}{r} = \frac{-2}{(-\frac{1}{5})} = 10$$

Therefore, $a_1 = 10$ and $r = \frac{-1}{5}$.

Answer: $a_n = 10(-\frac{1}{5})^{n-1}$

Geometric Series

The sum of the terms of a geometric sequence constitutes a geometric series. To illustrate, the sum of the initial 5 terms of the geometric sequence characterized by $a_n = 3n + 1$ can be determined as follows:

$$S_5 = \sum_{n=1}^{5} 3^{n+1}$$

$$3^{1+1} + 3^{2+1} + 3^{3+1} + 3^{4+1} + 3^{5+1}$$

$$3^2 + 3^3 + 3^4 + 3^5 + 3^6$$

$$9 + 27 + 81 + 243 + 729$$

$$1{,}089$$

Adding 5 positive integers is easily manageable. However, computing the sum of a large number of terms becomes impractical. Therefore, we proceed to derive a formula for calculating the sum of the first n terms of any geometric sequence. In general,

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

Multiplying both sides by r we can write,

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Subtracting these two equations we then obtain,

$$S_n - rS_n = a_1 - a_1 r^n$$

$$S_n(1 - r) = a_1(1 - r^n)$$

Assuming $r \neq 1$dividing both sides by $(1-r)$ leads us to the formula for the **n^{th} partial sum of a geometric sequence**:

$$S_n = \frac{a_1(1-r^n)}{1-r} (r \neq 1)$$

In other words, the nth partial sum of any geometric sequence can be computed using the first term and the common ratio. For instance, to find the sum of the first 15 terms of the geometric sequence defined by $a_n = 3n + 1$, apply the formula with $a_1 = 9$ and $r = 3$.

$$S_{15} = \frac{a_1(1-r^{15})}{1-r} = \frac{9 \cdot (1-3^{15})}{1-?}$$

$$= \frac{9(-14,348,906)}{-2} = 64,570,077$$

Ex.4 Determine the sum of the initial 10 terms in the provided sequence: 4, −8, 16, −32, 64,…

Sol. Check if there is a consistent ratio between the given terms.

$$r = -\frac{8}{4} = -2$$

The sequence is a geometric sequence with a common ratio of -2. Utilize the common ratio (-2) and the first term ($a_1 = 4$) to find the sum of the initial 10 terms.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{4[1-(-2)^{10}]}{1-(-2)} = \frac{4(1-1,024)}{1+2}$$

$$\frac{4(-1,023)}{3} = -1,364$$

Answer: $S_{10}=-1,364$

Ex.5 Evaluate: $\sum_{n=1}^{6} 2(-5)^n$

Sol. Here, we need to determine the sum of the first 6 terms of a geometric sequence with the general term an = $2(-5)^n$.

Utilize this information to find the first term and the common ratio r:

$$a_1=2(-5)^1=-10$$

To demonstrate the existence of a common ratio, we can express successive terms in general as follows:

$$r = \frac{a_n}{a_n} = \frac{2(-5)^n}{2(-5)^{n-1}}$$

$$(-5)^{n-(n-1)}$$

$$(-5)^1 = -5$$

Use $a_1=-10$ and $r=-5$ to calculate the 6^{th} partial sum.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_6 = \frac{-10[1-(-5)^6]}{1-(-5)}$$

$$= \frac{-10(1-15,625)}{1+5}$$

$$= \frac{-10(-15,624)}{6} = 26,040$$

Answer: 26,040

If the common ratio r of an infinite geometric sequence is a fraction where $|r| < 1$ that is, $-1 < r < 1$, then the factor $1 - r^n$ in the formula for the nth partial sum approaches 1 as n increases.

For example, If r=110 and n=2,4,6

we have,
$$1-\left(\frac{1}{10}\right)^2 = 1 - 0.01 = 0.99$$
$$1-\left(\frac{1}{10}\right)^4 = 1 - 0.0001 = 0.9999$$
$$1-\left(\frac{1}{10}\right)^6 = 1 - 0.000001 = 0.999999$$

In this case, we observe that as n becomes larger, the factor approaches 1. This demonstrates the concept of a limit, a fundamental idea frequently utilized in advanced mathematics and denoted using the following notation:

$$\lim_{n\to\infty}(1-r^n) = 1 \text{ where } |r| < 1$$

This statement indicates that as n approaches infinity, the expression $1-r^n$ converges to 1. While this foreshadows advanced mathematical concepts, our current focus is on establishing a formula for specific infinite geometric series. Let's explore the n^{th} partial sum of a geometric sequence:

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}(1-r^n)$$

If $|r|<1|$ then the limit of the partial sums as n approaches infinity exists and we can write,

$$S_n = \frac{a_1}{1-r}(1-r^n) \underset{n\to\infty}{\Longrightarrow} S_\infty = \frac{a_1}{1-r}\cdot 1$$

Therefore, a convergent geometric series is an infinite geometric series where $|r|<1|$; its sum can be calculated using the formula:

$$S_\infty = \frac{a_1}{1-r}$$

Ex.6 Determine the sum of the infinite geometric series: $\frac{3}{2}+\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+\frac{1}{54}+\cdots$

Sol. Determine the common ratio, $r = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

Since the common ratio $r=\frac{1}{3}$ is a fraction between -1 and 1, this is a convergent geometric series. Use the first term $a_1=\frac{3}{2}$ and the common ratio to calculate its sum.

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{3}{2}}{1-(\frac{1}{3})}$$

$$= \frac{\frac{3}{2}}{\frac{2}{3}} = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4}$$

Answer: $S_\infty = \frac{9}{4}$

Note: In the scenario of an infinite geometric series where $|r| \geq 1|$, the series diverges, and there is no sum.

For example, if $a_n = (5)^{n-1}$ then r=5 and

We have $S_\infty = \sum_{n=1}^{\infty}(5)^{n-1} = 1 + 5 + 25 + \cdots$

We observe that this sum increases indefinitely and does not converge to a specific value.

A recurring decimal can be expressed as an infinite geometric series with a common ratio being a power of $\frac{1}{10}$. Thus, the formula for a convergent geometric series can be applied to transform a repeating decimal into a fraction.

Ex.7 Express the recurring decimal 1.181818... as a fraction.

Sol. Identify the recurring digits to the right of the decimal and express them as a geometric progression.0.181818…=0.18+0.0018+0.000018+…

$$= \frac{18}{100} + \frac{18}{10,000} + \frac{18}{1,000,000} + \cdots$$

In this form we can determine the common ratio,

$$r = \frac{\frac{18}{10,000}}{\frac{18}{100}} = \frac{18}{10.000} \times \frac{100}{18} = \frac{1}{100}$$

Note that the ratio between any two successive terms is $\frac{1}{100}$.

Use this and the fact that $a_1 = \frac{18}{100}$ to calculate the infinite sum:

$$S_\infty = \frac{a_1}{1-r}$$

$$= \frac{\frac{18}{100}}{1-(\frac{1}{100})} = \frac{\frac{18}{100}}{\frac{99}{100}}$$

$$= \frac{18}{100}\cdot\frac{100}{99} = \frac{2}{11}$$

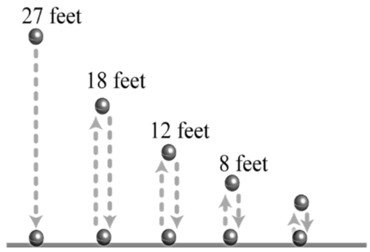
Therefore, $0.181818... = \frac{2}{11}$ and

We have, $1.181818\ldots = 1 + \frac{2}{11} = 1\frac{2}{11}$

Answer: $1\frac{2}{11}$

Ex.8 A specific ball rebounds to two-thirds of the height from which it fell. If the ball is initially dropped from a height of 27 feet, estimate the total distance the ball travels.

Sol. We can calculate the height of each successive bounce:



$27 \cdot \frac{2}{3} = 18$ feet Height of the first bounce

$18 \cdot \frac{2}{3} = 12$ feet Height of the second bounce

$12 \cdot \frac{2}{3} = 8$ feet Height of the third bounce

The overall distance covered by the ball is the sum of the distances during its descent and ascent. The descent forms a geometric series, 27+18+12+⋯ Distance the ball is falling

Where, $a_1 = 27$ and $r = \frac{2}{3}$.

As the common ratio (r) is a fraction between -1 and 1, we can compute this sum as follows:

$$S_\infty = \frac{a_1}{1-r}$$

$$= \frac{27}{1-\frac{2}{3}} = \frac{27}{\frac{1}{3}} = 81$$

Hence, the ball descends a cumulative distance of 81 feet. The upward movements of the ball constitute a geometric series,

18+12+8+⋯ Distance the ball is rising

Where, $a_1 = 18$ and $r = \frac{2}{3}$.

Calculate this sum in a similar manner:

$$S_\infty = \frac{a_1}{1-r} = \frac{18}{1-\frac{2}{3}}$$

$$= \frac{18}{\frac{1}{3}} = 54$$

So, the ball ascends a total distance of 54 feet. Estimate the overall distance covered by summing the total upward and downward distances:

81+54=135 feet

Answer: 135 feet

NOTE:

A geometric sequence is a sequence in which the ratio (r) between successive terms remains constant.

The general term of a geometric sequence can be expressed in terms of its first term (a_1), common ratio (r), and index (n) as follows:

$$a_n = a_1 r^{n-1}.$$

A geometric series is the summation of the terms in a geometric sequence.

The n^{th} partial sum of a geometric sequence can be determined using the initial term a1 and the common ratio r, as expressed below:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

The sum of an infinite geometric sequence can be computed when the common ratio is a fraction between -1 and 1 (i.e., $|r| < 1$), as indicated below:

$$S_\infty = \frac{a_1}{1-r}. \text{ If } |r| \geq 1 \text{ Then no sum exists.}$$