BINOMIAL THEOREM

GENERAL TERM AND MIDDLE TERM

Pascal's Triangle

A triangular arrangement of numbers is presented, depicting coefficients for the expansion of $(x + y)^n$. The initial row corresponds to n = 0, the second to n = 1, and so forth. Each row commences and concludes with 1. Intermediate numbers are derived by summing the two adjacent numbers in the row above.

$$\begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \\ 1\quad 4\quad 6\quad 4\quad 1 \\ 1\quad 5\quad 10\quad 10\quad 5\quad 1 \\ 1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1 \\ \text{etc.} \end{array}$$

IMPORTANT TERMS IN THE BINOMIAL EXPANSION

(a) General Term:

The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r} y^r$$

Ex.1 Find

(i) 28^{th} term of $(5x + 8y)^{30}$

(ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Sol.

(i) $$T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27}$$

$$= \frac{30!}{3!27!}\,(5x)^3 \,.\, (8y)^{27}$$

(ii) $$7\text{th term of } \left(\frac{4x}{5} - \frac{5}{2x}\right)^9$$

$$T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6$$

$$= \frac{9!}{3!6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6$$

$$= \frac{10500}{x^3}$$

(b) **Middle Term:** The term or terms at the center of the expansion of $(x + y)^n$ is (are):

(i) If n is even, there exists only one central term, and it is determined by:

$$T_{\frac{(n+2)}{2}} = C^n_{\frac{n}{2}}.x^{\frac{n}{2}}.y^{\frac{n}{2}}$$

(ii) If n is odd, there are two central terms, which are:

$$T_{\frac{(n+1)}{2}} \ \& \ T_{\frac{(n+1)}{2}+1}$$

The central term possesses the highest binomial coefficient, and in the case of two central terms, their coefficients will be identical.

${}^nC_r$ will be maximum

When $r = \frac{n}{2}$ if n is even

When $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

In the expansion of $(1 + x)^n$, the term with the highest binomial coefficient will be the central term.

Ex.2 Determine the central term or terms in the expansion of

(i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(3a - \frac{a^3}{6}\right)^9$

Sol. (i) $\left(1 - \frac{x^2}{2}\right)^{14}$ (ii) $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term

$$T_s = C^{14}_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}$$

(ii) $$\left(3a - \frac{a^3}{6}\right)^9$$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2} + 1\right)^{th}$

It means T_5 & T_6 is middle terms

$$T_5 = C^9_4(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}a^{17}$$

$$T_6 = C^9_5(3a)^{9-5}(-\frac{a^3}{6})^5 = -\frac{21}{16}a^{19}$$

(c) Term Independent of x :

A term that is independent of x does not contain x. Therefore, determine the value of r for which the exponent of x is zero.

Ex.3 Find the term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$.

Sol. General term in the expansion is

$$C_r^{10}\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}}$$

$$C_r^{10}\, x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term,

$$r=\frac{10}{3}$$

Which is not an integer. Therefore, there will be no constant term.

(d) Numerically Greatest Term :

The binomial expansion of $(a+b)^n$ is expressed as follows: –

$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + ... + {}^nC_r\, a^{n-r}\, b^r + ... + {}^nC_n\, a^0 b^n$

By substituting specific values for a and b on the right-hand side (RHS), each term in the binomial expansion will assume a particular value. The term with the numerically highest value is referred to as the numerically greatest term.

T_r and T_{r+1} be the r^{th} and $(r+1)^{th}$ terms respectively

$$T_r = {}^nC_{r-1}\, a^{n-(r-1)}\, b^{r-1}$$

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\frac{a^{n-r}b^r}{a^{n-r+1}b^{r-1}}\right| = \frac{n-r+1}{r}.\left|\frac{b}{a}\right|$$

$$\left|\frac{T_{r+1}}{T_r}\right| \geq 1$$

$$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1$$

$$\frac{n+l}{r} - 1 \geq \left|\frac{a}{b}\right|$$

$$r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$$

Case-I: When $\frac{n+1}{1+|\frac{a}{b}|}$ is an integer (say m),

Then,

(i) $T_{r+1} > T_r$

When, $r < m$ ($r = 1, 2, 3 \ldots, m - 1$)

i.e. $T_2 > T_1, T_3 > T_2, \ldots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$

When, $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$

When, $r > m$ ($r = m + 1, m + 2, \ldots n$)

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II When $\frac{n+1}{1+|\frac{a}{b}|}$ is not an integer (Let its integral part be m),

then

(i) $T_{r+1} > T_r$

when $r < \frac{n+1}{1+|\frac{a}{b}|}$ ($r = 1, 2, 3, \ldots, m-1, m$)

i.e. $T_2 > T_1, T_3 > T_2, \ldots, T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$

when $r > \frac{n+1}{1+|\frac{a}{b}|}$ ($r = m + 1, m + 2, \ldots n$)

i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots, T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Notes:

(i) In any binomial expansion, the central term or terms possess the highest binomial coefficient.

In the expansion of $(a + b)^n$

n	No. of Greatest Binomial Coefficient	Greatest Binomial Coefficient
Even	1	${}^nC_{\frac{n}{2}}$
Odd	2	${}^nC_{\frac{(n-1)}{2}}$ and ${}^nC_{\frac{(n+1)}{2}}$ (Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put $a = b = 1$, and proceed as discussed above.

Ex.4 Determine the numerically greatest term in the expansion of $(3 - 5x)^{11}$ when x is equal to?

Sol. Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|} - 1 \le r \le \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

Solving we get $2 \le r \le 3$

$$r = 2, 3$$

So, the greatest terms are T_{2+1} and T_{3+1}.

Greatest term (when r = 2)

$$T_3 = {}^{11}C_2.3^9\,(-5x)^2 = 55.3^9 = T_4$$

From above we say that the value of both greatest terms are equal.

Ex.5 For a positive integer n, demonstrate that the integral part of $(7 + 4\sqrt{3})^n$is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ ....(i)

Where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now $0 < 7 - 4\sqrt{3} < 1$

$$0 < (7-4\sqrt{3})^n < 1$$

Let, $(7-4\sqrt{3})^n = f'$ .... (ii)

$$0<f'<1$$

Adding (i) and (ii)

$$I+ f+ f' = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$$

$$2 [\ {}^nC_0 7^n + {}^nC_2 7^{n-2} (4\sqrt{3})^2 + ...]$$

I+ f+ f' = even integer

(f +f' must be an integer)

$$0<f+ f'<2$$

$$f+ f'=1$$

I+1=even integer therefore I is an odd integer.

Ex.6 What is the remainder when dividing 5^{99} by 13?

Sol. $5^{99} = 5.5^{98} = 5.\ (25)^{49} = 5\ (26 - 1)^{49}$

$$5\ [{}^{49}C_0\ (26)^{49} - {}^{49}C_1\ (26)^{48} + ... + {}^{49}C_{48}\ (26)^1 - {}^{49}C_{49}\ (26)^0]$$

$$5\ [{}^{49}C_0\ (26)^{49} - {}^{49}C_1\ (26)^{48} + ...+ {}^{49}C_{48}\ (26)^1 - 1]$$

$$5\ [{}^{49}C_0\ (26)^{49} - {}^{49}C_1(26)^{48} + ... + {}^{49}C_{48}\ (26)^1 - 13] + 60$$

13 (k) + 52 + 8 (where k is a positive integer)

$$13\ (k + 4) + 8$$

Hence, remainder is 8.

Some Standard Expansions

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\ x^{n-r}\ y^r$$

$$= {}^nC_0\ x^n\ y^0 + {}^nC_1\ x^{n-1}\ y^1 + ...+ {}^nC_r\ x^{n-r}\ y^r + ...+ {}^nC_n\ x^0\ y^n\ \ ....(i)$$

(ii) Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r\ (-1)^r\ x^{n-r}\ y^r$$

$$= {}^nC_0\ x^n y^0 - {}^nC_1\ x^{n-1}\ y^1 + ...+ {}^nC_r (-1)^r\ x^{n-r} y^r + ...+ {}^nC_n\ (-1)^n x^0 y^n\ ....(ii)$$

(iii) Adding (i) & (ii), we get

$$(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n\, y^0 + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots\ldots]$$

(iv) Subtracting (ii) from (i), we get

$$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1}\, y^1 + {}^nC_3\, x^{n-3}\, y^3 + \ldots\ldots\ldots]$$

PROPERTIES OF BINOMIAL COEFFICIENTS

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots\ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \;;\; n \in N \quad \ldots.(i)$$

where $C_0, C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are called **combinatorial (Binomial) coefficients.**

(a) The sum of all the Binomial coefficients is 2^n.

Put $x = 1$, in (i)

we get $C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n = 2^n$

$$\Rightarrow \quad \sum_{r=0}^{n} {}^nC_r = 0 \quad \ldots.(ii)$$

(b) Put $x = -1$ in (i)

we get $C_0 - C_1 + C_2 - C_3 \ldots\ldots\ldots\ldots + C_n = 0$

$$\Rightarrow \quad \sum_{r=0}^{n} (-1)^{r\,n} C_r = 0 \quad \ldots(iii)$$

(c) The sum of the Binomial coefficients at odd position is equal to the sum of the Binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii),

$$C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots. = \dfrac{n(n-1)(n-2)\ldots.(n-r+1)}{r(r-1)(r-2)\ldots..1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.{}^{n+1}C_{r+1}$

Ex.7 Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots\ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Sol. LHS $^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow$ $^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow$ $^{12}C_{11} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow$ $^{13}C_{11} + {}^{13}C_{10} + \ldots\ldots\ {}^{25}C_{10}$

and so on. RHS = $^{26}C_{11}$

Alternative

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + \ldots\ldots\ldots\ldots (1+ x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{10} in $[(1 + x)^{26} - (1 + x)^{10}]$

$= {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Ex.8 Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + {}^nC_n = n \,.\, 2^{n-1}$

(ii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Sol. (i) L.H.S. $= \sum_{r=1}^{n} r.{}^nC_r = \sum_{r=1}^{n} r.\frac{n}{r}.{}^{n-1}C_{r-1}$

$$\sum_{r=1}^{n} r.{}^{n-1}C_{r-1} = n.[{}^{n-1}C_0 + {}^{n-1}C_0 + \ldots.. + {}^{n-1}C_{n-1}]$$

$$= n \,.\, 2^{n-1}$$

Aliter : (Using method of differentiation)

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n \quad \ldots\ldots\ldots(A)$$

Differentiating (A),

we get $n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + n.C_nx^{n-1}$.

Put $x = 1$,

$$C_1 + 2C_2 + 3C_3 + \cdots \ldots \ldots + n \cdot C_n = n \cdot 2^{n-1}$$

(ii) L.H.S. $= \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^{n} \frac{n+1}{r+1}\ {}^{n}C_r$

$$= \frac{1}{n+1} \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left[2^{n+1} - 1\right]$$

Aliter : (Using method of integration)

Integrating (A),

we get $\frac{(1+x)^{n+1}}{n+1} + C = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}$

(where C is a constant)

Put $x = 0$,

we get, $C = -\frac{1}{-n+1}$

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}$$

Put $x = 1$,

we get $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

Put $x = -1$,

we get $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots ... = \frac{1}{n+1}$

Ex.9 Prove that $C_1 - C_3 + C_5 - .... = 2^{\frac{n}{2}} \sin. \frac{n\pi}{4}$

Sol. Consider the expansion

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n \quad ....(i)$$

putting $x = -i$ in (i)

we get $(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + ...\ (-1)^n C_n i^n$

Or $2^{\frac{n}{2}}\left[\cos\left(-\frac{n\pi}{4}\right) + i\sin\left(-\frac{n\pi}{4}\right)\right]$

$$= (C_0 - C_2 + C_4 - \cdots ....) - i(C_1 - C_3 + C_5 - \cdots ...) \quad .....(ii)$$

Equating the imaginary part in (ii)

we get $C_1 - C_3 + C_5 - \cdots \ldots\ldots = 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$

Ex.10 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n$ then prove that

$$\sum_{0\le i<j\le n} \left(C_i + C_j\right)^2 = (n-1)^{2n}C_n + 2^{2n}$$

Sol. L.H.S $\displaystyle\sum_{0\le i<j\le n} \left(C_i + C_j\right)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \ldots. + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \ldots.$

$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \ldots + (C_2 + C_n)^2 + \ldots. + (C_{n-1} + C_n)^2$

$$= n(C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2) + 2\sum_{0\le i<j\le n} C_iC_j$$

$$= n^{2n}C_n + 2\cdot\left\{2^{2n-1} - \frac{2n!}{2\cdot n!\,n!}\right\}$$

$$n.\,{}^{2n}C_n + 2^{2n} - {}^{2n}C_n$$

$$= (n-1).{}^{2n}C_n + 2^{2n} = \text{R.H.S}$$