BINOMIAL THEOREM

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BINOMIAL THEOREM FOR POSITIVE INDEX

The Binomial Theorem is a mathematical formula enabling the expansion of any power of a binomial expression into a series. When dealing with a positive integer (n), the expansion is expressed as follows:

$$(a+x)^n$$

$$^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + {}^nC_nx^n$$

$$\sum_{r=0}^{n} {}^nC_ra^{n-r}x^r$$

Where $^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **Binomial co-efficient.**

Similarly

$$(a-x)^n = {}^nC_0a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 - \ldots + (-1)^{rn}C_ra^{n-r}x^r + \ldots + (-1)^{nn}C_nX^n$$

$$(a-x)^n = \sum_{r=0}^{n}(-1)^{rn}C_ra^{n-r}x^r$$

Replacing $a = 1$,

We get $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rX^r + \ldots + {}^nC_nX^n$

And $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2X^2 - \ldots + (-1)^{rn}C_rx^r + \ldots + (-1)^{nn}C_nx^n$

Observations:

- There exist (n+1) terms in the expansion of $(a + x)^n$.
- In the expansion of $(a + x)^n$, the sum of the powers of (x) and (a) in each term remains constant and is equal to (n).
- The general term in the expansion of $(a + x)^n$ is denoted by the (r+1)th term, and it is represented as:

$$T_{r+1} = {}^nC_r\mathbf{a}^{n-r}X^r$$

- The term located at position p from the end is equivalent to the term at position (n−p+2) from the beginning in the expansion of $(a + x)^n$.
- Coefficient of x^r in expansion of $(a + x)^n$ is $^nC_r\, a^{n-r}\, x^r$.

$$^nC_x = {}^nC_y$$

$$x = y \quad \text{or} \quad x + y = n.$$

In the expansion of $(a + x)^n$ and $(a - x)^n$, x^r occurs in $(r + 1)^{th}$ term.

Ex.1 If the coefficients corresponding to the second, third, and fourth terms in the expansion of the binomial expression are...

$(1 + x)^n$ are in A.P., show that $n = 7$.

Sol. According to the question ${}^nC_1 \cdot {}^nC_2 \cdot {}^nC_3$ are in A.P.

$$\frac{2n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$$n^2 - 9n + 14 = 0$$

$$\Rightarrow \quad (n-2)(n-7) = 0$$

$$\Rightarrow \quad n = 2 \text{ or } 7$$

Since, the symbol ${}^nC_3$ demands that n should be ≥ 3, n cannot be 2,

$$\therefore \quad n = 7 \text{ only.}$$

Ex.2 Find the

(i) last digit

(ii) last two digit

(iii) Last three digit of 17^{256}.

Sol.

$$(17)^{256} = (289)^{128} = (290 - 1)^{128}$$

$$\left[{}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + \ldots\ldots\ldots\right] + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000\ m + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000\ m + \frac{128 \times 127}{2} \times (290)^2 - \frac{128 \times 290}{1} + 1$$

$$= 1000\ m + 683527680 + 1$$

Hence, the last digit is 1. Last two digits is 81. Last three digit is 681.

Ex.3 If the binomial coefficients for the $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of

$(a + bx)^{18}$ are equal find r.

Sol. This is possible only when

Either $\quad 2r + 3 = r - 3 \quad$ ....(1)

Or $\quad 2r + 3 + r - 3 = 18 \quad$ ....(2)

From (1) $r = -6$ not possible but from (2) $r = 6$

Hence $r = 6$ is the only solution.

Ex.4 Determine the coefficient of

(i) x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$, (ii) and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Determine the relationship between a and b when these coefficients are equal.

Sol. The general term in $\left(ax^2 + \frac{1}{bx}\right)^{11} = {}^{11}C_r (ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$$= {}^{11}C_r \frac{a^{11-r}}{b^r} x^{22-3r}$$

If in this term power of x is 7, then 22 – 3r = 7

$\Rightarrow$ r = 5

$\therefore$ Coefficient of $x^7 = {}^{11}C_5 \frac{a^6}{b^5}$ ... (1)

The general term in $\left(ax - \frac{1}{bx^2}\right)^{11} = (-1)^r\, {}^{11}C_r (ax)^{11-r}\left(\frac{1}{bx^2}\right)^r$

$$(-1)^r\, {}^{11}C_r \frac{a^{11-r}}{b^r} x^{11-3r}$$

If in this term power of x is –7, then 11 – 3r = –7 $\Rightarrow$ r = 6

$$\text{Coefficient of } x^{-7} = (-1)^6\, {}^{11}C_6 \frac{a^{11-6}}{b^6} = {}^{11}C_5 \frac{a^5}{b^6}$$

If these two coefficient are equal,

$${}^{11}C_5 \frac{a^6}{b^5} = {}^{11}C_5 \frac{a^5}{b^6}$$

$$a^6b^6 = a^5b^5$$

$$a^5b^5(ab - 1) = 0$$

$$ab = 1 (a \neq 0,\ b \neq 0)$$