LIMITS AND DERIVATIVES

LIMITS

The limits Law

1 CONSTANT LAW

If f(x) equals a constant value c (making it a constant function), then

$$\lim_{x\to a} f(x) = \lim_{x\to a} C = C \qquad \text{...(i)}$$

2 SUM LAW

If both of the limits

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

exist, then

$$\lim_{x\to a}[f(x) \pm g(x)] = \left[\lim_{x\to a} f(x)\right] \pm \left[\lim_{x\to a} g(x)\right] = L \pm M \qquad \text{...(ii)}$$

(The limit of a sum is equivalent to the sum of the limits, and the limit of a difference is equal to the difference of the limits).

3 PRODUCT LAW

If both of the limits

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x)$$

exist, then

$$\lim_{x\to a}[f(x) g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM \qquad \text{...(iii)}$$

(The limit of a product corresponds to the product of the limits).

4 QUOTIENT LAW

If both of the limits

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

exist and if $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \qquad \text{...(iv)}$$

(The limit of a quotient is equal to the quotient of the limits, as long as the limit of the denominator is not zero.)

5 ROOT LAW

For a positive integer n and when $a > 0$ for even values of n, then .

$$\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a} \qquad \ldots(v)$$

The root law is self-evident for the case when $n = 1$.

$$\lim_{x\to a} x = a$$

Examples (ii) and (iii) illustrate how the limit laws can be applied to determine the limits of polynomials and rational functions.

Ex.1 $\lim_{x\to 3}\left(x^2+2x+4\right)$

Sol.

$$\lim_{x\to 3}\left(x^2+2x+4\right)$$

$$=\left(\lim_{x\to 3}x^2\right)+\left(\lim_{x\to 3}2x\right)+\left(\lim_{x\to 3}4\right)$$

$$=\left(\lim_{x\to 3}x\right)^2+2\left(\lim_{x\to 3}x\right)+\left(\lim_{x\to 3}4\right)$$

$$=3^2+2.3+4=19$$

Ex.2 $\lim_{x\to 3}\frac{2x+5}{x^2+2x+4}$

Sol.

$$\lim_{x\to 3}\frac{2x+5}{x^2+2x+4}=\frac{\lim_{x\to 3}(2x+5)}{\lim_{x\to 3}\left(x^2+2x+4\right)}$$

$$=\frac{2.3+5}{3^2+2.3+4}=\frac{11}{19}$$

Note: In examples 1 and 2, we methodically utilized the limit laws until we could directly substitute 3 for $\lim_{x\to 3}$ x in the last step. When calculating the limit of a quotient involving polynomials, it is imperative to confirm, before reaching this final step, that the limit of the denominator is not zero. If the limit of the denominator equals zero, then the overall limit may not exist.

6 SOME IMPORTANT FORMULAE

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1$ (ii) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(iii) $\lim\limits_{x\to 0}\frac{\sin^{-1}x}{x}=1$ (iv) $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}=1$

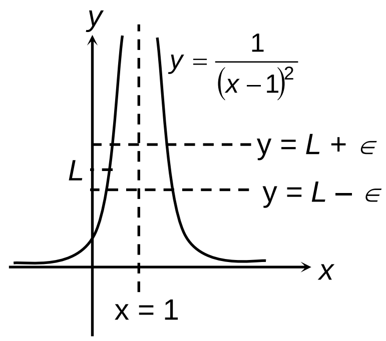
(v) $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$ (vi) $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a$

(vii) $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ (viii) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(ix) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ (x) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

Ex.3 Investigate $\lim\limits_{x\to 1}\frac{1}{(x-1)^2}$

Sol. Since, $\lim\limits_{x\to 1}(x-1)^2=0$ the quotient law cannot be applied.



Moreover, we can make $\frac{1}{(x-1)^2}$ arbitrarily large be choosing x sufficiently close to 1.

Therefore, $\frac{1}{(x-1)^2}$ cannot approach any finite value L as x approaches 1.

Therefore, the limit in this example does not exist. You can see the geometric reason if you examine the graph of $y=\frac{1}{(x-1)^2}$ in figure.

As x approaches 1, the point (x, y) moves along the curve close to the vertical line x = 1. Consequently, it moves out of the defined strip bounded by the two horizontal lines $x=L-\in$ and $x=L+\in$ that enclose the suggested limit L. Hence, the point (x, y) cannot converge towards the point (1, L) as x approaches 1.

Ex.4 Investigate $\lim\limits_{x\to 2}\frac{x^2-4}{x^2+x-6}$

Sol. We cannot directly employ the quotient law due to the denominator approaching zero as x approaches 2.

If the numerator were approaching a value other than zero as x → 2, the limit would not exist (as in Example 1). However, in this case, the numerator does tend towards zero, leaving the possibility that a factor in the numerator can cancel out the same factor in the denominator, thereby resolving the issue of a zero denominator. In fact,

$$\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+3)}=\lim_{x\to 2}\frac{x+2}{x+3}=\frac{4}{5}$$

We are able to eliminate the factor x - 2 since it is not zero: $x \neq 2$ when we compute the limit as x approaches 2. Furthermore, this confirms the numerical limit of 0.8 that we obtained in the provided limit.