LIMITS AND DERIVATIVES

LIMITS OF TRIGONOMETRIC FUNCTIONS

Limits of trigonometric Functions

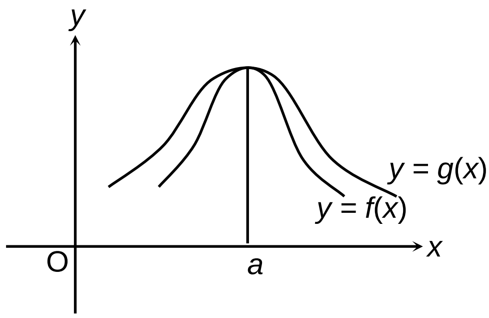
The following general facts about functions can be quite useful when calculating limits for certain trigonometric functions.

Theorem 1

Consider two real-valued functions, f and g, defined over the same domain, where $f(x) \le g(x)$ for all x within the domain of definition.

For some a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

This is illustrated in figure:



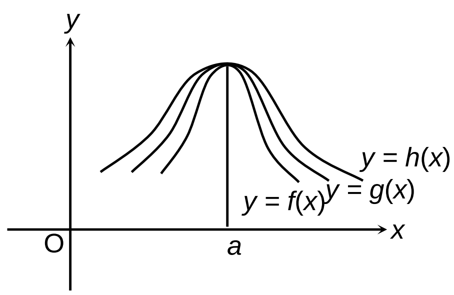
Theorem 2 (Sandwich Theorem)

Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real member a,

if $$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x),$$

then $$\lim_{x \to a} g(x) = 1.$$

This is illustrated in figure:

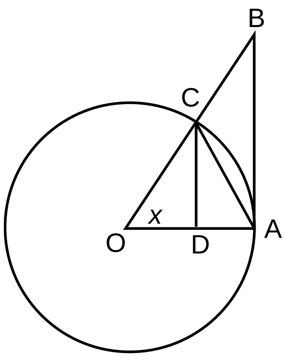


Here is a geometric demonstration of the following significant inequality involving trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \text{ for } 0 < |x| < \frac{\pi}{2} \qquad \text{...(i)}$$

Proof: We are aware that sin (–x) and cos (–x) = cos x. Therefore, it is adequate to establish the inequality for $0 < x < \frac{\pi}{2}$. In the illustration, O is the centre of the unit circle, with angle AOC measuring x radians, and $0 < x < \frac{\pi}{2}$. Perpendicular line segments BA and CD are drawn to OA. Additionally, we connect AC.

Area of ΔOAC < Area of sector OAC < Area of ΔOAB



i.e. $$\frac{1}{2}OA.CD < \frac{x}{2\pi}.\pi.(OA)^2 < \frac{1}{2}OA.AB$$

i.e. $$CD < x.OA < AB$$

From ΔOCD $$\sin x = \frac{CD}{OA}$$

(since OC = OA) and hence CD = OA sin x

Also, $$\tan x = \frac{AB}{OA}$$

and $$AB = OA.\tan x$$

Thus, $$OA \sin x < OA.x < OA.\tan x$$

Since length OA is positive,

we have $$\sin x < x < \tan x$$

Since $0 < x < \frac{\pi}{2}$, sin x is positive and thus by dividing throughout by sin x,

we have $$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Taking reciprocals throughout, we have $\cos x < \frac{\sin x}{x} < 1$, which complete the proof.

Now, $$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} 1$$

$\Rightarrow$ $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Ex.1 Evaluate: $\lim_{x\to 2} \frac{x-3}{x+4}$

Sol. $$\lim_{x\to 2} \frac{x-3}{x+4} = \frac{2-3}{2+4} = -\frac{1}{6}$$

Ex.2 Evaluate: $\lim_{x\to 2} \frac{x^5-32}{x^3-8}$

Sol. $$\lim_{x\to 2} \frac{x^5-32}{x^3-8} = \lim_{x\to 2} \left(\frac{x^5-32}{x-2}\right) \div \left(\frac{x^3-8}{x-2}\right)$$

$$= \lim_{x\to 2} \left(\frac{x^5-2^5}{x-2}\right) \div \lim_{x\to 2} \left(\frac{x^3-2^3}{x-2}\right)$$

$$= 5(2)^4 \div 3(2)^2 = \frac{20}{3}$$

$$\left[\text{As } \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

Ex.3 Evaluate: $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

Sol. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$, we rationalize i.e. multiply and divide by $\left(\sqrt{1+x}+\sqrt{1-x}\right)$

$$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x[\sqrt{1+x}+\sqrt{1-x}]}$$

$$= \lim_{x \to 0} \frac{2x}{x[\sqrt{1+x} + \sqrt{1-x}]}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0} + \sqrt{1-0}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1$$

Ex.4 If $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$ and $n \in I^+$, find n.

Sol. $$\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$$

$\Rightarrow$ $$n.2^{n-1} = 5.2^4$$

$\Rightarrow$ $$n = 5$$

Ex.5 Evaluate: $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$.

Sol. $$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin\frac{ax+bx}{2}.\sin\frac{bx-ax}{2}}{x^2}$$

$$\lim_{x \to 0} 2.\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}} \times \frac{a+b}{2}.\frac{\sin\frac{(b-a)x}{2}}{\frac{(b-a)x}{2}} \times \frac{(b-a)}{2}$$

$$= 2 \times \frac{a+b}{2} \times \frac{b-a}{2} = \frac{b^2 - a^2}{2}$$

Ex.6 Evaluate: $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$.

Sol. $$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

Put $x - \frac{\pi}{4} = t$,

as $x \to \frac{\pi}{4}$, $t \to 0$

$$= \lim_{t \to 0} \frac{\sin\left(\frac{\pi}{4} + t\right) - \cos\left(\frac{\pi}{4} + t\right)}{t}$$

$$= \lim_{t \to 0} \frac{\sin\frac{\pi}{4}\cos t + \cos\frac{\pi}{4}\sin t - \cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t}{t}$$

$$= \lim_{t \to 0} \frac{\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t - \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t}{t}$$

$$= \frac{2}{\sqrt{2}} \lim_{t \to 0} \frac{\sin t}{t} = \sqrt{2} \times 1 = \sqrt{2}$$