LIMITS AND DERIVATIVES

INTRODUCTION OF LIMITS AND DERIVATIVES

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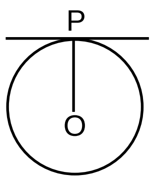
In such scenarios, we express that $\lim_{x \to n} f(x)$ is undefined. The limit in (i) is referred to as the left-hand limit (L.H.L.) at x = n, and the limit in (ii) is termed the right-hand limit (R.H.L.) at x = n.

Tangent line and slope predictor (For competitive exam)

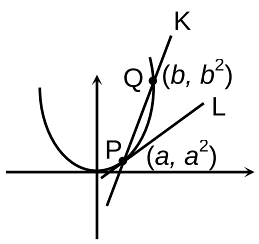
In elementary geometry, the tangent line to a point P on a circle is described as the straight line passing through P, and it is perpendicular to the radius extending from the centre O to point P.



In the context of a general graph represented as y = f(x), there is no radius to work with. However, the tangent line to the graph at point P should be the straight line passing through P, aligning with the curve's direction at P. The direction of a line is, in essence, defined by its slope.

To illustrate this, let's consider an example:

(a) Calculate the slope of the tangent line L to the parabola $y = x^2$ at the point (a, a^2)



We cannot directly compute the slope of L as we have the coordinates of only one point (a, a^2) on L. Therefore, we commence by working with another line for which we can calculate the slope.

In the diagram, there's a secant line K that goes through point P and a neighboring point Q (b, b^2) on the parabola $y = x^2$. We'll denote h as Δx, representing the difference between the x-coordinate of P and Q. (The symbol Δx

signifies an increment or change in the x-value.) The coordinates of point Q are expressed as follows:

$$b = a + h \text{ and } b^2 = (a + h)^2$$

As a result, the disparity in the y-coordinates of P and Q is.

$$\Delta y = b^2 - a^2 = (a + h)^2 - a^2$$

Since P and Q represent distinct points, we can employ the slope definition to compute the slope m_{PQ} of the secant line K that passes through P and Q. If you alter the value of $h = \Delta x$, the line K also changes, and as a result, its slope m_{PQ} is contingent on the value of h.

$$m_{PQ} = \frac{\Delta y}{\Delta x}$$

$$= \frac{(a+h)^2 - a^2}{(a+h)-a} = \left(\frac{2ah + h^2}{h}\right) = \frac{h(2a+h)}{h}$$

Since h is a nonzero value, we can eliminate it,

and the expression becomes $= (2a + h)$.

Now, as point Q moves progressively closer to point P along the curve, the line K continues to pass through both P and Q. When Q is in close proximity to P, h approaches zero, and the secant line K closely approximates the tangent line L.

The concept here is to define the tangent line L as the limiting position of the secant line K.

h: approaches zero

Q: approaches P, and so

K: approaches L, mean while

The slope of K gradually becomes equivalent to the slope of L as h approaches zero.

As the value of h tends towards zero, we can inquire about the values towards which the slope $m_{PQ} = 2a + h$ converges. We can frame this inquiry about the "limiting value" of $2a + h$ by expressing it as:

$$\lim_{h \to 0}(2a + h)$$

In this context, "lim" is a shortened form of the word "limit," and "$h \to 0$" is an abbreviation for the statement "h approaches zero." Consequently, we can respond by determining the limit of $2a + h$ as h approaches zero.

For a specific value of a, we can empirically explore this question by computing values of $2a + h$ as h approaches zero. We can use values of h that progressively approach zero, such as $h = -0.1$, $h = -0.01$, $h = -0.001$, $h = -0.0001$, and so on, or $h = 0.5$, $h = 0.1$, $h = 0.05$, $h = 0.01$, and so forth.

As an example, the tables of values (shown in the figure) reveal that when a = 2 and a = –4, we should deduce that $\lim_{h\to 0}(2+h)=2$ and $\lim_{h\to 0}(-4+h)=-4$

In a more general sense, it becomes evident from the table in the figure that

$$\lim_{h\to 0} m_{PQ} = \lim_{h\to 0}(2a+h) = 2a \qquad ...(i)$$

h	2 + h
0.1	2.1
0.01	2.01
0.001	2.001
0.0001	2.0001
↓	↓
0	2

h	–4 + h
0.5	–3.5
0.1	–3.9
0.05	–3.95
0.01	–3.99
0.005	–3.995
0.001	–3.999
↓	↓
0	–4

h	2a + h
0.01	2a + 0.01
0.001	2a + 0.001
!	!
↓	↓
0	2a

As $h \to 0$ (first column), 2 + h approaches 2 (second column)

As $h \to 0$ (first column), –4 + h approaches –4 (second column)

As $h \to 0$ (first column), 2a + h approaches 2a (second column)

In conclusion, this provides a response to our initial query: The slope $m = m_{PQ}$ of the tangent line to the parabola $y = x^2$ at the point (a, a^2) can be expressed as follows:

$$m = 2a \qquad ...(ii)$$

The equation in (ii) serves as a "slope predictor" for the tangent lines to the parabola $y = x^2$. Once we have knowledge of the slope of the tangent line at a specific point on the curve, we can subsequently employ the point-slope formula to formulate an equation for this tangent line.

We establish the slope m of the tangent line to the graph $y = f(x)$ at the point

$P(a, f(a))$ as:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $(a + h)$ corresponds to $(h = x - a)$, we establish that as h approaches 0, x tends towards a.

So, $$m = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$