CONIC SECTIONS

INTRODUCTION OF CONIC SECTIONS

Introduction of Conic Sections -

When a plane cuts the cone other than the vertex, we have the following situations:

(a) When $\beta = 90°$, the section is a circle

(b) When $\alpha < \beta < 90°$, the section is an ellipse

(c) When $\alpha = \beta$; the section is a parabola

(d) When $0 \le \beta < \alpha$; the section is a hyperbola

Where β is the angle made by the plane with the vertical axis of the cone.

Circle:

Set of points in a plane equidistant from a fixed point. A circle with radius r and center (h, k) can be represented as $(x - h)^2 + (y - k)^2 = r^2$

Parabola:

Set of points in a plane that are equidistant from a fixed-line and point. A parabola with $a > 0$, focus at (a, 0), and directix $x = -a$ can be represented as $y^2 = 4ax$

In parabola $y^2 = 4ax$, the length of the latus rectum is given by 4a.

Ellipse:

The sum of distances of a set of points in a plane from two fixed points is constant. An ellipse with foci on the x-axis can be represented as:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The length of the latus rectum is given by $\frac{2b^2}{a}$

Hyperbola:

The difference of distances of set of points in a plane from two fixed points is constant. The hyperbola with foci on the x-axis can be represented as:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

In a Hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The length of latus rectum is given by $\frac{2b^2}{a}$

Ex 1. Find an equation of the circle with center at (0, 0) and radius r.

Sol. Given, Centre = (h, k) = (0, 0)

Radius = r

Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Ex 2. Find the equation of the circle with center (–3, 2) and radius 4.

Sol. Given, Centre = (h, k) = (-3, 2)

Radius = r = 4

Therefore, the equation of the required circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x + 3)^2 + (y - 2)^2 = 4^2$$

$$(x + 3)^2 + (y - 2)^2 = 16$$

Ex. 3 Find the center and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Sol. The given equation is

$$x^2 + y^2 + 8x + 10y - 8 = 0$$

$$(x^2 + 8x) + (y^2 + 10y) = 8$$

By completing the squares within the parenthesis, we get

$$(x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$$

i.e. $(x + 4)^2 + (y + 5)^2 = 49$

i.e. $[x - (-4)]^2 + [y - (-5)]^2 = 7^2$

Comparing with the standard form, $h = -4$, $k = -5$ and $r = 7$

Therefore, the given circle has center at $(-4, -5)$ and radius 7.

Ex 4. Find the equation of the circle which passes through the points $(2, -2)$, and $(3, 4)$ and whose center lies on the line $x + y = 2$.

Sol. Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Given that the circle passes through the points $(2, -2)$ and $(3, 4)$.

Thus, $(2 - h)^2 + (-2 - k)^2 = r^2$ ....(1)

and $(3 - h)^2 + (4 - k)^2 = r^2$ ....(2)

Also, given that the center lies on the line $x + y = 2$.

$\Rightarrow$ $h + k = 2$ ....(3)

Solving the equations (1), (2) and (3), we get

$h = 0.7$, $k = 1.3$ and $r^2 = 12.58$

Hence, the equation of the required circle is

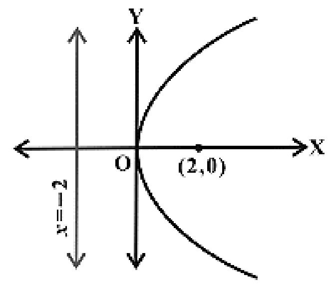
$$(x - 0.7)^2 + (y - 1.3)^2 = 12.58$$

Ex 5. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

Sol. The given equation involves y^2, that means the axis of symmetry is along the x-axis.

The coefficient of x is positive so the parabola opens to the right.

Now by comparing the given equation with $y^2 = 4ax$, $a = 2$



Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is $x = -2$ (see the figure)

Length of the latus rectum $= 4a = 4 \times 2 = 8$

Ex. 6: Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2).

Sol. Given, Vertex = (0,0)

Focus = (0,2)

The focus lies on the y-axis.

Thus, the y-axis is the axis of the parabola.

Therefore, the equation of the parabola is of the form $x^2 = 4ay$.

$\Rightarrow \quad x^2 = 4(2)y$

$\Rightarrow \quad x^2 = 8y$

This is the required equation of parabola.

Ex. 7 Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Sol. Given, $9x^2 + 4y^2 = 36$

The given equation of the ellipse can be written in standard form as:

$$\left(\frac{x^2}{4}\right)+\left(\frac{y^2}{9}\right)=1$$

Here, 9 is greater than 4.

Thus, the major axis is along the y-axis.

By comparing with $\left(\frac{x^2}{b^2}\right)+\left(\frac{y^2}{a^2}\right)=1,$

$$a^2 = 9, \quad b^2 = 4$$

$\Rightarrow$ $a = 3, \quad b = 2$

$$c=\sqrt{\left(a^2-b^2\right)}=\sqrt{\left(9-4\right)}=\sqrt{5}$$

$$e=\frac{c}{a}=\frac{\sqrt{5}}{3}$$

Therefore, the foci are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, vertices are (0, 3) and (0, –3), length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is $\frac{\sqrt{5}}{3}$.

Ex. 8 Find the equation of the hyperbola whose foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Sol. Given foci are $(0, \pm 12)$

That means $c = 12$.

Length of the latus rectum $=\frac{2b^2}{a}=36$

or $b^2 = 18a$

Now, $c^2 = a^2 + b^2$

$$144 = a^2 + 18a$$

i.e., $a^2 + 18a - 144 = 0$

$\Rightarrow$ $a = -24, 6$

The value of a cannot be negative.

Therefore, $a = 6$ and so $b^2 = 108$.

Hence, the equation of the required hyperbola is $\left(\frac{y^2}{36}\right) - \left(\frac{x^2}{108}\right) = 1$ or $3y^2 - x^2 = 108$.