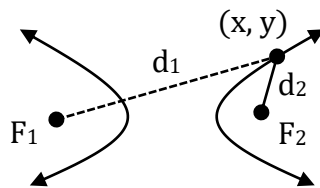


CONIC SECTIONS

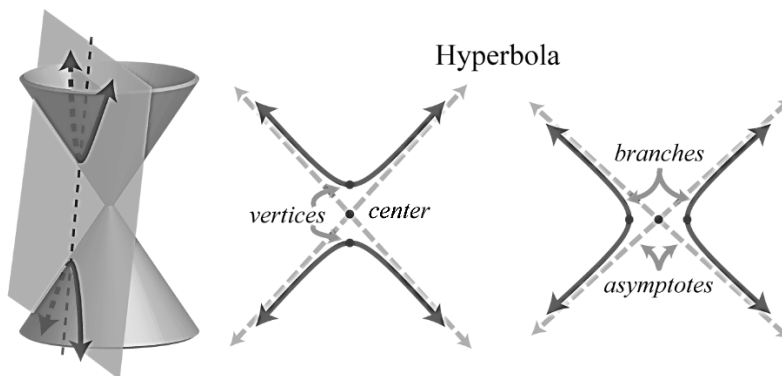
HYPERBOLAS

The Hyperbola in Standard Form

A **hyperbola** is the set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci and d is some given positive constant then (x, y) is a point on the hyperbola if $d = |d_1 - d_2|$ as pictured below:

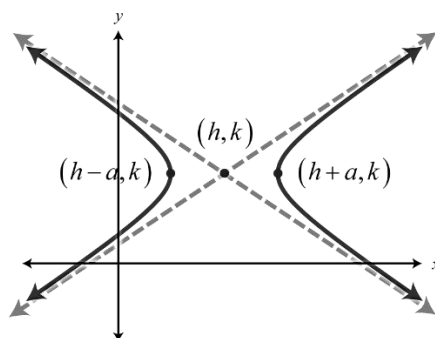


In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called **branches**. Points on the separate branches of the graph where the distance is at a minimum are called **vertices**. The midpoint between a hyperbola's vertices is its center. Unlike a parabola, a hyperbola is asymptotic to certain lines drawn through the center. In this section, we will focus on graphing hyperbolas that open left and right or upward and downward.



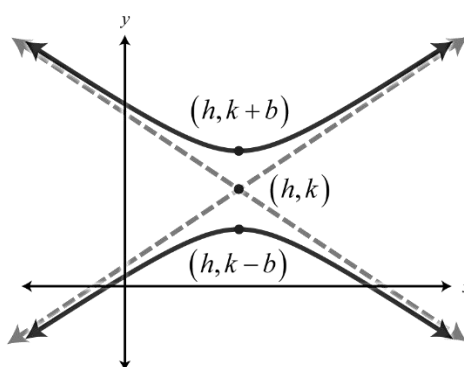
The asymptotes are drawn dashed as they are not part of the graph; they simply indicate the end behavior of the graph. The equation of a **hyperbola opening left and right in standard form** follows:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



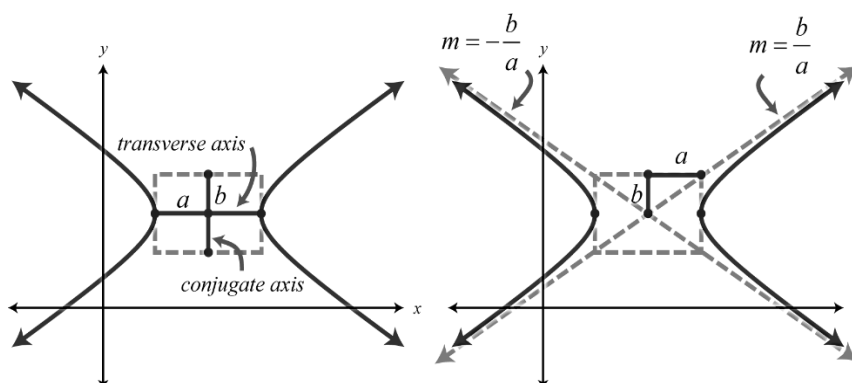
Here the center is (h, k) and the vertices are $(h \pm a, k)$. The equation of a hyperbola opening upward and downward in standard form follows:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



Here the center is (h, k) and the vertices are $(h, k \pm b)$.

The asymptotes are essential for determining the shape of any hyperbola. Given standard form, the asymptotes are lines passing through the center (h, k) with slope $m = \pm \frac{b}{a}$. To easily sketch the asymptotes we make use of two special line segments through the center using a and b . Given any hyperbola, the **transverse axis** is the line segment formed by its vertices. The **conjugate axis** is the line segment through the center perpendicular to the **transverse axis** as pictured below:



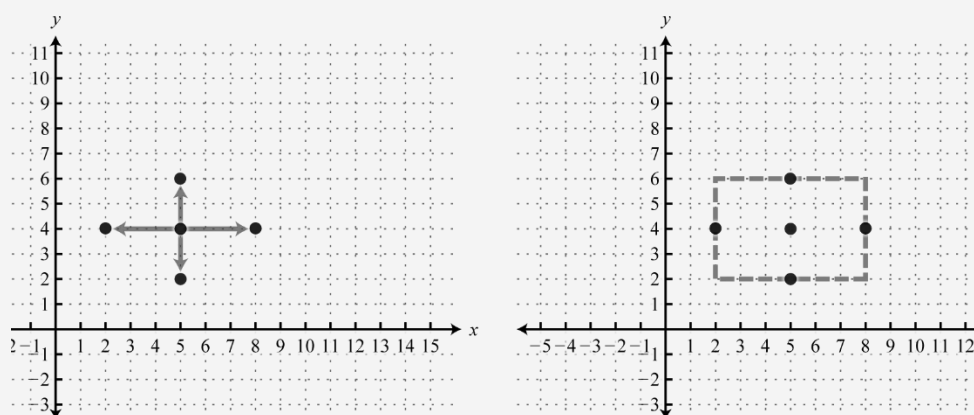
The rectangle defined by the transverse and conjugate axes is called the **fundamental rectangle**. The lines through the corners of this rectangle have slopes $m = \pm \frac{b}{a}$. These lines are the asymptotes that define the shape of the hyperbola. Therefore, given standard form, many of the properties of a hyperbola are apparent.

Equation	Centre	a	b	Opens
$\frac{(x-3)^2}{25} - \frac{(y-5)^2}{16} = 1$	(3,5)	a=5	b=4	Left and right
$\frac{(y-2)^2}{36} - \frac{(x+1)^2}{9} = 1$	(-1,2)	a=3	b=6	Upward and downward
$\frac{(y+2)^2}{3} - (x-5)^2 = 1$	(5,-2)	a=1	b= $\sqrt{3}$	Upward and downward
$\frac{x^2}{49} - \frac{(y+4)^2}{8} = 1$	(0,-4)	a=7	b= $2\sqrt{2}$	Left and right

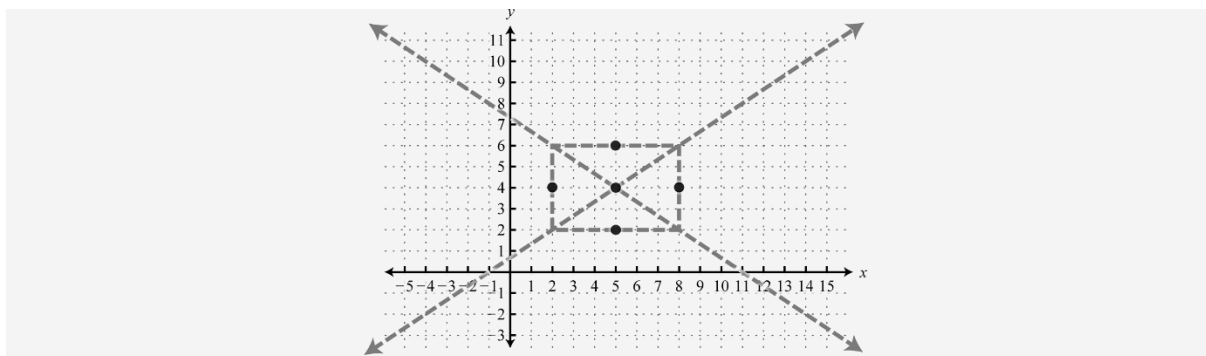
The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.

Ex.1 Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1$

Sol. In this case, the expression involving x has a positive leading coefficient; therefore, the hyperbola opens left and right. Here $a = \sqrt{9} = 3$ and $b = \sqrt{4} = 2$. From the center (5,4), mark points 3 units left and right as well as 2 units up and down. Connect these points with a rectangle as follows:

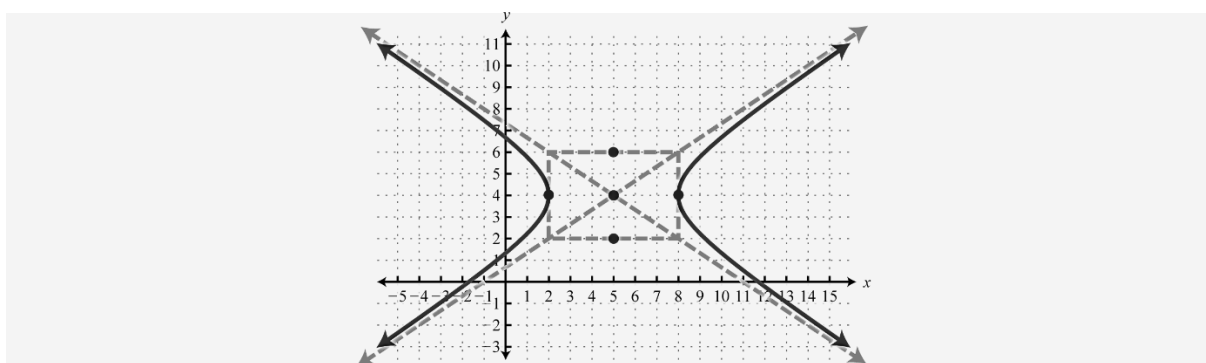


The lines through the corners of this rectangle define the asymptotes.



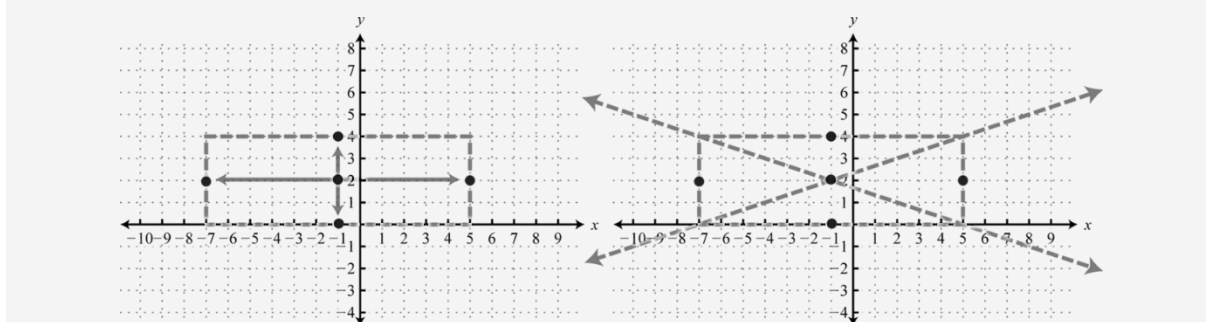
Use these dashed lines as a guide to graph the hyperbola opening left and right passing through the vertices.

Answer:



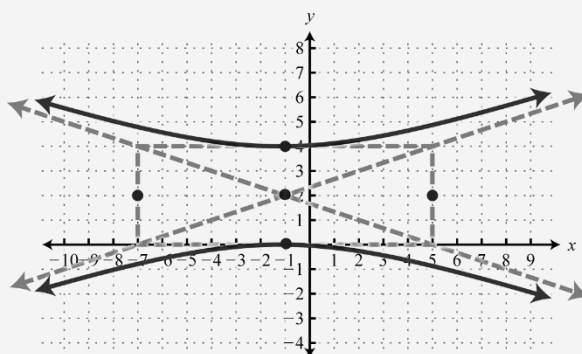
Ex.2 Graph: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$.

Sol. In this case, the expression involving y has a positive leading coefficient; therefore, the hyperbola opens upward and downward. Here $a = \sqrt{36} = 6$ and $b = \sqrt{4} = 2$. From the center $(-1, 2)$ mark points 6 units left and right as well as 2 units up and down. Connect these points with a rectangle. The lines through the corners of this rectangle define the asymptotes.



Use these dashed lines as a guide to graph the hyperbola opening upward and downward passing through the vertices.

Answer:



Note:

When given a hyperbola opening upward and downward, as in the previous example, it is a common error to interchange the values for the center, h and k . This is the case because the quantity involving the variable y usually appears first in standard form. Take care to ensure that the y -value of the center comes from the quantity involving the variable y and that the x -value of the center is obtained from the quantity involving the variable x .

As with any graph, we are interested in finding the x - and y -intercepts.

Ex.3 Find the intercepts: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$

Sol. To find the x -intercepts set $y=0$ and solve for x .

$$\frac{(0-2)^2}{4} - \frac{(x+1)^2}{36} = 1$$

$$1 - \frac{(x+1)^2}{36} = 1$$

$$-\frac{(x+1)^2}{36} = 0$$

$$(x+1)^2 = 0$$

$$x + 1 = 0$$

$$x = -1$$

Therefore there is only one x-intercept, $(-1,0)$. To find the y-intercept set $x=0$ and solve for y.

$$\frac{(y-2)^2}{4} - \frac{(0+1)^2}{36} = 1$$

$$\frac{(y-2)^2}{4} - \frac{1}{36} = 1$$

$$\frac{(y-2)^2}{4} = \frac{37}{36}$$

$$\frac{(y-2)}{2} = \pm \frac{\sqrt{37}}{6}$$

$$y-2 = \pm \frac{\sqrt{37}}{3}$$

$$y = 2 \pm \frac{\sqrt{37}}{3} = \frac{6 \pm \sqrt{37}}{3}$$

Therefore there are two y-intercepts, $\left(0, \frac{6-\sqrt{37}}{3}\right) \approx (0, -0.03)$ and

$\left(0, \frac{6+\sqrt{37}}{3}\right) \approx (0, 4.03)$. Take a moment to compare these to the sketch of the graph in the previous example.

Answer: x-intercept: $(-1,0)$; y-intercepts: $\left(0, \frac{6-\sqrt{37}}{3}\right)$ and $\left(0, \frac{6+\sqrt{37}}{3}\right)$

Consider the hyperbola centered at the origin,

$$9x^2 - 5y^2 = 45$$

Standard form requires one side to be equal to 1. In this case, we can obtain standard form by dividing both sides by 45.

$$\frac{9x^2 - 5y^2}{45} = \frac{45}{45}$$

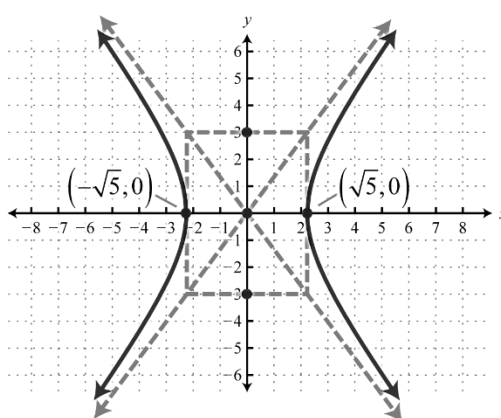
$$\frac{9x^2}{45} - \frac{5y^2}{45} = \frac{45}{45}$$

$$\frac{x^2}{5} - \frac{y^2}{9} = 1$$

This can be written as follows:

$$\frac{(x-0)^2}{5} - \frac{(y-0)^2}{9} = 1$$

In this form, it is clear that the center is $(0,0)$, $a=\sqrt{5}$, and $b=3$. The graph follows.



The Hyperbola in General Form

We have seen that the graph of a hyperbola is completely determined by its center, vertices, and asymptotes; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **hyperbola in general form** follows:

$$px^2 - qy^2 + cx + dy + e = 0 \quad \text{Hyperbola opens left and right.}$$

$$qy^2 - px^2 + cx + dy + e = 0 \quad \text{Hyperbola opens upward and downward.}$$

where $p, q > 0$. The steps for graphing a hyperbola given its equation in general form are outlined in the following example.

Ex.4 Graph: $4x^2 - 9y^2 + 32x - 54y - 53 = 0$.

Sol. Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$4x^2 - 9y^2 + 32x - 54y - 53 = 0$$

$$(4x^2 + 32x + \dots) + (-9y^2 - 54y + \dots) = 53$$

$$4(x^2 + 8x + \dots) - 9(y^2 + 6y + \dots) = 53$$

Step 2: Complete the square for each grouping. In this case, for the terms

involving x use $\left(\frac{8}{2}\right)^2 = (4)^2 = 16$ and for the terms involving y use $\left(\frac{6}{2}\right)^2 = (3)^2 = 9$

The factor in front of each grouping affects the value used to balance the equation on the right,

$$4(x^2 + 8x + 16) - 9(y^2 + 6y + 9) = 53 + 64 - 81$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $4 \cdot 16 = 64$. Similarly, adding 9 inside of the second grouping is equivalent to adding $-9 \cdot 9 = -81$. Now factor and then divide to obtain 1 on the right side.

$$4(x + 4)^2 - 9(y + 3)^2 = 36$$

$$\frac{4(x + 4)^2 - 9(y + 3)^2}{36} = \frac{36}{36}$$

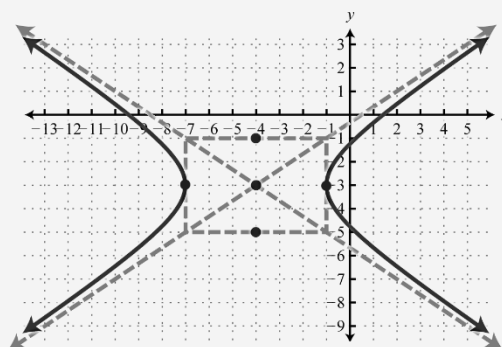
$$\frac{4(x + 4)^2}{36} - \frac{9(y + 3)^2}{36} = \frac{36}{36}$$

$$\frac{(x + 4)^2}{9} - \frac{(y + 3)^2}{4} = 1$$

Step 3: Determine the center, a , and b , and then use this information to sketch the graph.

In this case, the center is $(-4, -3)$, $a = \sqrt{9} = 3$, and $b = \sqrt{4} = 2$. Because the leading coefficient of the expression involving x is positive and the coefficient of the expression involving y is negative, we graph a hyperbola opening left and right.

Answer:



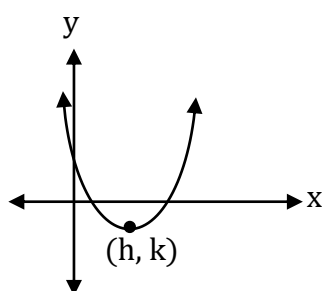
Identifying the Conic Sections

In this section, the challenge is to identify a conic section given its equation in general form. To distinguish between the conic sections, use the exponents and coefficients. If the equation is quadratic in only one variable and linear in the other, then its graph will be a parabola.

Parabola: $a > 0$

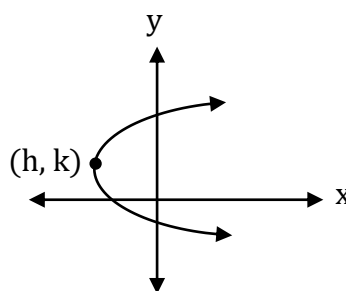
$$y = a(x - h)^2 + k$$

$$y = ax^2 + bx + c$$



$$x = a(y - k)^2 + h$$

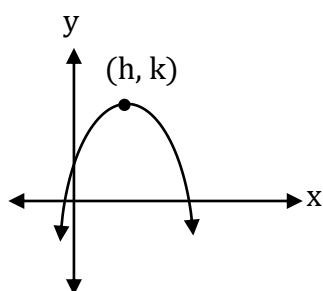
$$x = ay^2 + by + c$$



Parabola: $a < 0$

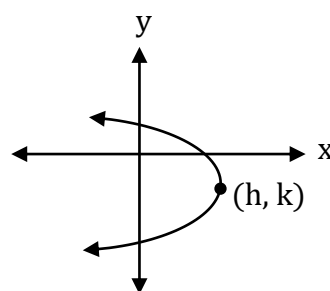
$$y = a(x - h)^2 + k$$

$$y = ax^2 + bx + c$$



$$x = a(y - k)^2 + h$$

$$x = ay^2 + by + c$$

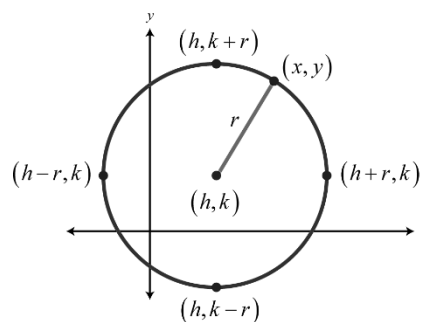


If the equation is quadratic in both variables, where the coefficients of the squared terms are the same, then its graph will be a circle.

Circle: $(x-h)^2 + (y-k)^2 = r^2$

$$x^2 + y^2 + cx + dy + e = 0$$

If the equation is quadratic in both variables where the coefficients of the squared terms are different but have the same sign, then its graph will be an ellipse.

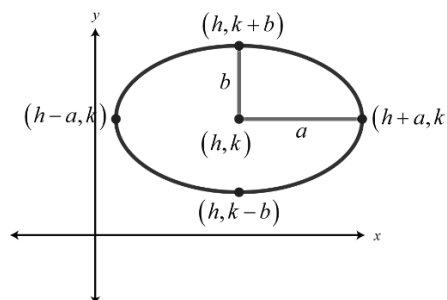


Ellipse: $a, b > 0$ and $p, q > 0$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$px^2 + qy^2 + cx + dy + e = 0$$

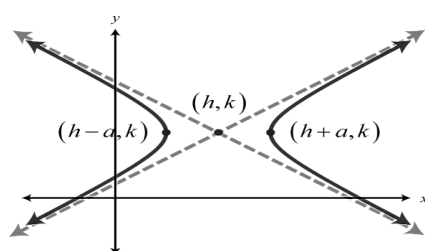
If the equation is quadratic in both variables where the coefficients of the squared terms have different signs, then its graph will be a hyperbola.



Hyperbola: $a, b > 0$ and $p, q > 0$

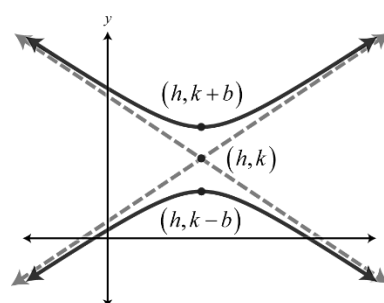
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$px^2 - qy^2 + cx + dy + e = 0$$



$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

$$qy^2 - px^2 + cx + dy + e = 0$$



Ex.5 Identify the graph of each equation as a parabola, circle, ellipse, or hyperbola.

(a) $4x^2 + 4y^2 - 1 = 0$

(b) $3x^2 - 2y^2 - 12 = 0$

(c) $x - y^2 - 6y + 11 = 0$

Sol. (a) The equation is quadratic in both x and y where the leading coefficients for both variables is the same, 4.

$$4x^2 + 4y^2 - 1 = 0$$

$$4x^2 + 4y^2 = 1$$

$$x^2 + y^2 = \frac{1}{4}$$

This is an equation of a circle centered at the origin with radius $\frac{1}{2}$.

(b) The equation is quadratic in both x and y where the leading coefficients for both variables have different signs.

$$3x^2 - 2y^2 - 12 = 0$$

$$\frac{3x^2 - 2y^2}{12} = \frac{12}{12}$$

$$\frac{x^2}{4} - \frac{y^2}{6} = 1$$

This is an equation of a hyperbola opening left and right centered at the origin.

(c) The equation is quadratic in y only.

$$x - y^2 + 6y - 11 = 0$$

$$x = y^2 - 6y + 11$$

$$x = (y^2 - 6y + 9) + 11 - 9$$

$$x = (y - 3)^2 + 2$$

This is an equation of a parabola opening right with vertex (2,3).

Answer: (a) Circle

(b) Hyperbola

(c) Parabola

Notes:

- The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.
- The center, vertices, and asymptotes are apparent if the equation of a hyperbola is given in standard form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$

- To graph a hyperbola, mark points a units left and right from the center and points b units up and down from the center. Use these points to draw the fundamental rectangle; the lines through the corners of this rectangle are the asymptotes. If the coefficient of x^2 is positive, draw the branches of the hyperbola opening left and right through the points determined by a . If the coefficient of y^2 is positive, draw the branches of the hyperbola opening up and down through the points determined by b .
- The orientation of the transverse axis depends the coefficient of x^2 and y^2 .
- If the equation of a hyperbola is given in general form $px^2 - qy^2 + cx + dy + e = 0$ or $qy^2 - px^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings to obtain standard form.
- We recognize the equation of a hyperbola if it is quadratic in both x and y where the coefficients of the squared terms are opposite in sign.