CONIC SECTIONS

ELLIPSES

The Ellipse in Standard Form

An **ellipse** is the set of points in a plane whose distances from two fixed points, called foci, have a sum that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci (plural of focus) and d is some given positive constant then (x, y) is a point on the ellipse if $d=d_1+d_2$ as pictured below:



In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. Points on this oval shape where the distance between them is at a maximum are called **vertices** and define the **major axis**. The center of an ellipse is the midpoint between the vertices. The **minor axis** is the line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is at a minimum. The endpoints of the minor axis are called **co-vertices**.



If the major axis of an ellipse is parallel to the x-axis in a rectangular coordinate plane, we say that the ellipse is horizontal. If the major axis is parallel to the y-axis, we say that the ellipse is vertical. In this section, we are only concerned with sketching these two types of ellipses. However, the ellipse has many real-world applications and further research on this rich subject is encouraged. In a rectangular coordinate plane, where the center of a horizontal ellipse is (h, k), we have



As pictured a>b where a, one-half of the length of the major axis, is called the **major radius**. And b, one-half of the length of the **minor axis**, is called the minor radius. The equation of an **ellipse in standard form** follows:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The vertices are $(h \pm a, k)$ and $(h, k \pm b)$ and the orientation depends on a and b. If a>b, then the ellipse is horizontal as shown above and if a<b, then the ellipse is vertical and b becomes the major radius. What do you think happens when a=b?

| Equation | Center | а | b | Orientation |
|--|--------|--------------|---------------|-------------|
| $\frac{(x-1)^2}{4} + \frac{(y-8)^2}{9} = 1$ | (1,8) | a=2 | b=3 | Vertical |
| $\frac{(x-3)^2}{2} + \frac{(y+5)^2}{16} = 1$ | (3,-5) | $a=\sqrt{2}$ | b=4 | Vertical |
| $\frac{(x+1)^2}{1} + \frac{(y-7)^2}{8} = 1$ | (-1,7) | a=1 | $b=2\sqrt{2}$ | Vertical |
| $\frac{x^2}{25} + \frac{(y+6)^2}{10} = 1$ | (0,-6) | a=5 | b=√10 | Horizontal |

The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius, all of which can be determined from its equation written in standard from.

Ex.1 Graph:
$$\frac{(x+3)^2}{4} + \frac{(y-2)^2}{25} = 1$$

Sol. Written in this form we can see that the center of the ellipse is (-3, 2),

a = $\sqrt{4}$ = 2, and b = $\sqrt{25}$ = 5. From the center mark points 2 units to the left and right and 5 units up and down.



As with any graph, we are interested in finding the x- and y-intercepts.

Ex.2 Find the intercepts:
$$\frac{(x+3)^2}{4} + \frac{(y-2)^2}{25} = 1.$$

Sol. To find the x-intercepts set y=0:

$$\frac{(x+3)^2}{4} + \frac{(0-2)^2}{25} = 1$$
$$\frac{(x+3)^2}{4} + \frac{4}{25} = 1$$
$$\frac{(x+3)^2}{4} = 1 - \frac{4}{25}$$
$$\frac{(x+3)^2}{4} = \frac{21}{25}$$

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At this point we extract the root by applying the square root property.

$$\frac{x+3}{2} = \pm \sqrt{\frac{21}{25}}$$
$$x+3 = \pm \frac{2\sqrt{21}}{5}$$
$$= -3 \pm \frac{2\sqrt{21}}{5} = \frac{-15 \pm 2\sqrt{21}}{5}$$

Setting x=0 and solving for y leads to complex solutions, therefore, there are no y-intercepts. This is left as an exercise.

Answer: x-intercepts = $\left(\frac{-15 \pm 2\sqrt{21}}{5}, 0\right)$; y-intercepts = none.

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Unlike a circle, standard form for an ellipse requires a 1 on one side of its equation.

- **Ex.3** Graph and label the intercepts: $(x-2)^2+9(y-1)^2=9$.
- **Sol.** To obtain standard form, with 1 on the right side, divide both sides by 9.

$$\frac{(x-2)^2 + 9(y-1)^2}{9} = \frac{9}{9}$$
$$\frac{(x-2)^2}{9} + \frac{9(y-1)^2}{9} = \frac{9}{9}$$
$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

Therefore, the center of the ellipse is (2,1), $a = \sqrt{9} = 3$, and $b = \sqrt{1} = 1$. The graph follows:



To find the intercepts we can use the standard form $\frac{(x-2)^2}{9} + (y-1)^2 = 1$

x - intercepts set y = 0

$$\frac{(x-2)^2}{9} + (0-1)^2 = 1$$
$$\frac{(x-2)^2}{9} + 1 = 1$$
$$(x-2)^2 = 0$$
$$x - 2 = 0$$
$$x = 2$$

y - intercepts set x = 0

$$\frac{(0-2)^2}{9} + (y-1)^2 = 1$$
$$\frac{4}{9} + (y-1)^2 = 1$$
$$(y-1)^2 = \frac{5}{9}$$
$$y-1 = \pm \sqrt{\frac{5}{9}}$$
$$y = 1 \pm \frac{\sqrt{5}}{3} = \frac{3 \pm \sqrt{5}}{3}$$

MATHS

MATHS

Therefore the x-intercept is (2, 0) and the y-intercepts are

$$\left(0,\frac{3+\sqrt{5}}{3}\right)$$
 and $\left(0,\frac{3-\sqrt{5}}{3}\right)$.

Answer:



Consider the ellipse centered at the origin,

$$x^2 + \frac{y^2}{4} = 1$$

Given this equation we can write,

$$\frac{(x-0)^2}{1^2} + \frac{(y-0)^2}{2^2} = 1$$

In this form, it is clear that the center is (0, 0), a=1, and b=2. Furthermore, if we solve for y we obtain two functions:

$$x^{2} + \frac{y^{2}}{4} = 1.$$
$$\frac{y^{2}}{4} = 1 - x^{2}$$
$$y^{2} = 4(1 - x^{2})$$
$$y = \pm \sqrt{4(1 - x^{2})}$$
$$y = \pm 2\sqrt{1 - x^{2}}$$

The function defined by $y = 2\sqrt{1-x^2}$ is the top half of the ellipse and the function defined by $y = -2\sqrt{1-x^2}$ is the bottom half.



The Ellipse in General Form

We have seen that the graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of an ellipse in general form follows,

$$px^2 + qy^2 + cx + dy + e = 0$$

where p, q > 0. The steps for graphing an ellipse given its equation in general form are outlined in the following example.

Ex.4 Graph: 2x²+9y²+16x-90y+239=0

Sol. Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$2x^{2} + 9y^{2} + 16x - 90y + 239 = 0$$

(2x² + 16x + ...)+(9y² - 90y +) = -239
2(x² + 8x + ...)+9(y² - 10y + ...) = -239

Step 2: Complete the square for each grouping. In this case, for the terms involving x

Use
$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$
 and for the terms involving y use $\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$.

The factor in front of the grouping affects the value used to balance the equation on the right side:

 $2(x^2 + 8x + 16) + 9(y^2 - 10y + 25) = -239 + 32 + 225$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $2 \times 16 = 32$. Similarly, adding 25 inside of the second grouping is equivalent to adding $9 \times 25 = 225$.

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$$2(x+4)^{2} + 9(y-5)^{2} = 18$$

$$\frac{2(x+4)^{2} + 9(y-5)^{2}}{18} = \frac{18}{18}$$

$$\frac{2(x+4)^{2}}{18} + \frac{9(y-5)^{2}}{18} = \frac{18}{18}$$

$$\frac{(x+4)^{2}}{9} + \frac{(y-5)^{2}}{2} = 1$$

- **Step 3:** Determine the center, a, and b. In this case, the center is (-4,5), $a = \sqrt{9} = 3$, and $b = \sqrt{2}$.
- **Step 4:** Use a to mark the vertices left and right of the center, use b to mark the vertices up and down from the center, and then sketch the graph. In this case, the vertices along the minor axes $(-4, 5 \pm \sqrt{2})$ are not apparent and should be labeled.



Answer:



- **Ex.5** Determine the center of the ellipse as well as the lengths of the major and minor axes: $5x^2+y^2-3x+40=0$.
- **Sol.** In this example, we only need to complete the square for the terms involving x.

$$5x^2 + y^2 - 30x + 40 = 0$$

MATHS

$$5(x^2-6x+...)+y^2=-40$$

Use $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ for the first grouping to be balanced by 5×9=45 on the right side.

$$5(x^{2}-6x+9)+y^{2} = -40+45$$

$$5(x-3)^{2}+y^{2} = 5$$

$$\frac{5(x-3)^{2}+y^{2}}{5} = \frac{5}{5}$$

$$\frac{(x-3)^{2}}{1} + \frac{y^{2}}{5} = 1$$

Here, the center is (3, 0) $a = \sqrt{1} = 1$, and $b = \sqrt{5}$. Because b is larger than a, the length of the major axis is 2b and the length of the minor axis is 2a.

Answer: Center: (3,0) major axis: $2\sqrt{5}$ units ; minor axis: 2 units.

Notes:

- The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius.
- The center, orientation, major radius, and minor radius are apparent if the equation of an ellipse is given in standard form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$
- To graph an ellipse, mark points a units left and right from the center and points b units up and down from the center. Draw an ellipse through these points.
- The orientation of an ellipse is determined by a and b. If a>b then the ellipse is wider than it is tall and is considered to be a horizontal ellipse. If a
b then the ellipse is taller than it is wide and is considered to be a vertical ellipse.
- If the equation of an ellipse is given in general form $px^2+qy^2+cx+dy+e=0$ where p, q>0, group the terms with the same variables, and complete the square for both groupings.
- We recognize the equation of an ellipse if it is quadratic in both x and y and the coefficients of each square term have the same sign.