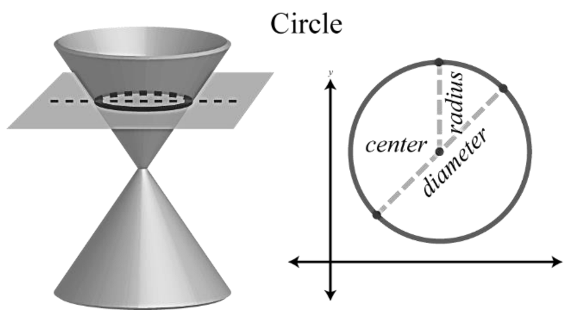
CONIC SECTIONS

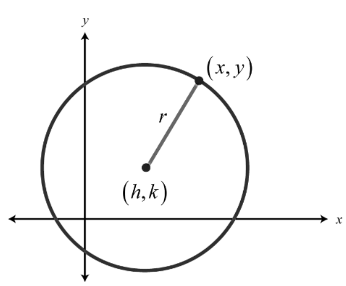
CIRCLE

The Circle in Standard Form

A **circle** is the set of points in a plane that lie a fixed distance, called the **radius**, from any point, called the center. The **diameter** is the length of a line segment passing through the center whose endpoints are on the circle. In addition, a circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone:



In a rectangular coordinate plane, where the center of a circle with radius *r* is (h, k), we have



Calculate the distance between (h, k) and (x, y) using the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Squaring both sides leads us to the equation of a **circle in standard form,**

$$(x-h)^2+(y-k)^2=r^2$$

In this form, the center and radius are apparent.

For example, given the equation $(x-2)^2+(y+5)^2=16$

we have, $$(x-h)^2+(x-k)^2=r^2$$

$$(x-2)^2+[y-(-5)]^2=4^2$$

In this case, the center is (2, −5)(2,−5) and r = 4.

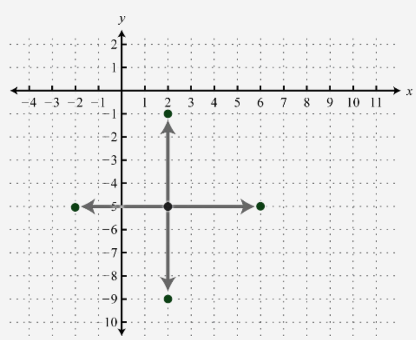
More examples follow:

Equation	Center	Radius
$(x-3)^2+(y-4)^2=25$	(3,4)	r = 5
$(x-1)^2+(y+2)^2=7$	(1,−2)	$r=\sqrt{7}$
$(x+4)^2+(y-3)^2=1$	(−4,3)	r =1
$x^2+(y+6)^2=8$	(0,−6)	$r=2\sqrt{2}$

The graph of a circle is completely determined by its center and radius.

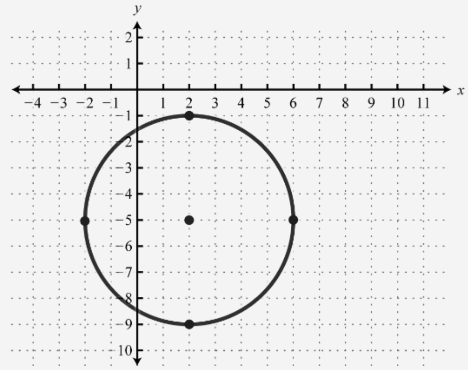
Ex. 1 Graph: $(x-2)^2+(y+5)^2=16$.

Sol. Written in this form we can see that the center is (2,−5) and that the radius r = 4 units. From the center mark points 4 units up and down as well as 4 units left and right.



Then draw in the circle through these four points.

Answer:



As with any graph, we are interested in finding the x- and y-intercepts.

Ex. 2 Find the intercepts: $(x-2)^2+(y+5)^2 = 16$.

Sol. To find the y-intercepts set $x = 0$:

$$(x-2)^2+(y+5)^2=16$$

$$(0-2)^2+(y+5)^2=16$$

$$4+(y+5)^2=16$$

For this equation, we can solve by extracting square roots.

$$(y+5)^2 = 12$$

$$y+5 = \pm\sqrt{12}$$

$$y+5 = \pm 2\sqrt{3}$$

$$y = -5 \pm 2\sqrt{3}$$

Therefore, the y-intercepts are $(0, -5-2\sqrt{3})$ and $(0, -5+2\sqrt{3})$.

To find the x-intercepts set $y=0$:

$$(x - 2)^2 + (y + 5)^2 = 16$$

$$(x - 2)^2 + (0 + 5)^2 = 16$$

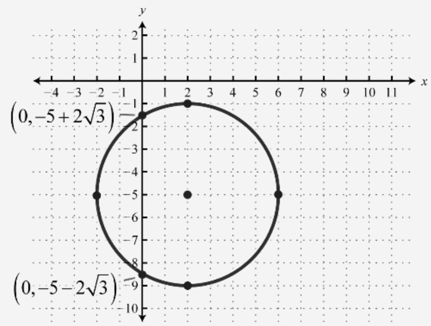
$$(x - 2)^2 + 25 = 16$$

$$(x - 2)^2 = -9$$

$$x - 2 = \pm\sqrt{-9}$$

$$x = 2 \pm 3i$$

And because the solutions are complex we conclude that there are no real x-intercepts. Note that this does make sense given the graph.

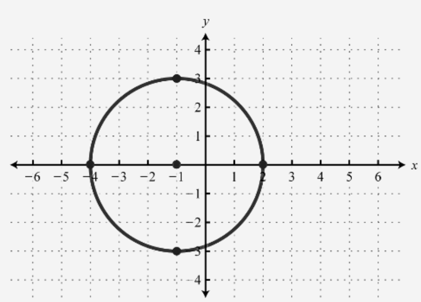


Answer: x-intercepts = none; y-intercepts = $(0, -5-2\sqrt{3})$ and $(0, -5+2\sqrt{3})$

Given the center and radius of a circle, we can find its equation.

Ex. 3 Graph the circle with radius r = 3units centered at (−1, 0). Give its equation in standard form and determine the intercepts.

Sol. Given that the center is (−1,0) and the radius is r = 3 we sketch the graph as follows:



Substitute h, k, and r to find the equation in standard form.

Since (h, k) = (−1,0) and r = 3

we have,
$$(x-h)^2+(y-k)^2=r^2$$
$$[x-(-1)]^2+(y-0)^2=3^2$$
$$(x+1)^2+y^2=9$$

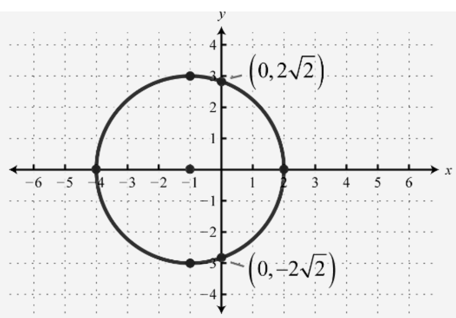
The equation of the circle is $(x+1)^2+y^2=9$, use this to determine the y-intercepts.

$$(x+1)^2+y^2=9$$

Set $x = 0$ to and solve for y.

$$(0+1)^2+y^2=9$$
$$1+y^2=9$$
$$y^2=8$$
$$y=\pm\sqrt{8}$$
$$y=\pm\, 2\sqrt{2}$$

Therefore, the y-intercepts are $(0, -2\sqrt{2})$ and $(0, 2\sqrt{2})$. To find the x-intercepts algebraically, set $y = 0$and solve for x; this is left for the reader as an exercise.



Answer: Equation = $(x+1)^2+y^2=9$; y-intercepts = $(0,-2\sqrt{2})$ and $(0,2\sqrt{2})$;

x-intercepts = $(-4,0)$ and $(2,0)$

Of particular importance is the **unit circle**,

$$x^2+y^2=1$$

Or, $(x-0)^2+(y-0)^2=1^2$

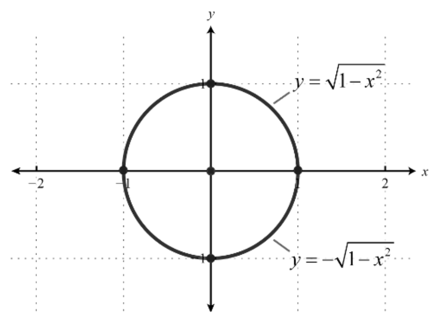
In this form, it should be clear that the center is (0,0)and that the radius is 1 unit. Furthermore, if we solve for y we obtain two functions:

$$x^2+y^2=1$$

$$y^2=1-x^2$$

$$y=\pm\sqrt{1-x^2}$$

The function defined by $y=\sqrt{1-x^2}$ is the top half of the circle and the function defined by $y=-\sqrt{1-x^2}$ is the bottom half of the unit circle:



The Circle in General Form

We have seen that the graph of a circle is completely determined by the center and radius which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **circle in general form** follows:

$$x^2+y^2+cx+dy+e=0$$

Here c, d, and e are real numbers. The steps for graphing a circle given its equation in general form follow.

Ex.4 Graph: $x^2+y^2+6x-8y+13=0$

Sol. Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. In this case, subtract 13 on both sides and group the terms involving x and the terms involving y as follows.

$$x^2 + y^2 + 6x - 8y + 13 = 0$$

$$(x^2 + 6x + \_\_\_) + (y^2 - 8y + \_\_\_) = -13$$

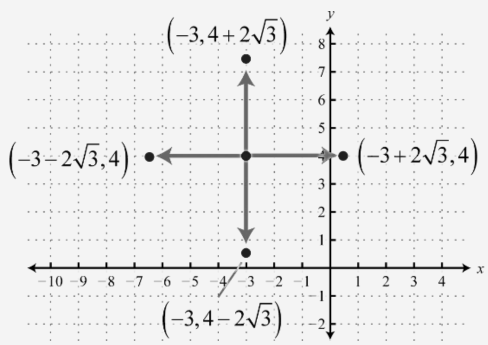
Step 2: Complete the square for each grouping. The idea is to add the value that completes the square, $\left(\frac{b}{2}\right)^2$, to both sides for both groupings, and then factor. For the terms involving x use $\left(\frac{6}{2}\right)^2 = 3^2 = 9$ and for the terms involving y use $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = -13 + 9 + 16$$

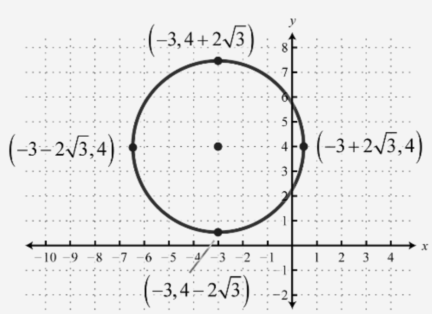
$$(x+3)^2 + (y-4)^2 = 12$$

Step 3: Determine the center and radius from the equation in standard form. In this case, the center is (−3, 4) and the radius $r = \sqrt{12} = 2\sqrt{3}$.

Step 4: From the center, mark the radius vertically and horizontally and then sketch the circle through these points.



Answer:



Ex. 5 Determine the center and radius: $4x^2+4y^2-8x+12y-3 = 0$.

Sol. We can obtain the general form by first dividing both sides by 4.

$$\frac{4x^2+4y^2-8x+12y-3}{4}=\frac{0}{4}$$

$$x^2+y^2-2x+3y-\frac{3}{4}=0$$

Now that we have the general form for a circle, where both terms of degree two have a leading coefficient of 1, we can use the steps for rewriting it in standard form. Begin by adding $\frac{3}{4}$ to both sides and group variables that are the same.

$$\left(x^2-2x+\_\right)+\left(y^2+3y+\_\right)=\frac{3}{4}$$

Next complete the square for both groupings. Use $\left(\frac{-2}{2}\right)^2=(-1)^2=1$ for the first grouping and $\left(\frac{3}{2}\right)^2=\frac{9}{4}$ for the second grouping.

$$\left(x^2-2x+1\right)+\left(y^2+3y+\frac{9}{4}\right)=\frac{3}{4}+1+\frac{9}{4}$$

$$(x-1)^2+\left(y+\frac{3}{2}\right)^2=\frac{16}{4}$$

$$(x-1)^2+\left(y+\frac{3}{2}\right)^2=4$$

Answer: Center $=\left(1,-\frac{3}{2}\right)$ radius = r = 2

In summary, to convert from standard form to general form we multiply, and to convert from general form to standard form we complete the square.

Standard form

$(x+3)^2 + (y-4)^2 = 12$

Complete the square ↺ Multiply ↻

$x^2 + y^2 + 6x - 8y + 13 = 0$

General form

Notes:

- The graph of a circle is completely determined by its center and radius.
- Standard form for the equation of a circle is $(x-h)^2+(y-k)^2=r^2$. The center is (h,k) and the radius measures r units.
- To graph a circle mark points r units up, down, left, and right from the center. Draw a circle through these four points.
- If the equation of a circle is given in general form $x^2+y^2+cx+dy+e=0$, group the terms with the same variables, and complete the square for both groupings. This will result in standard form, from which we can read the circle's center and radius.
- We recognize the equation of a circle if it is quadratic in both x and y where the coefficient of the squared terms are the same.