QUADRILATERALS

THEOREM RELATED TO QUADRILATERAL

PARALLELOGRAM

A parallelogram is a quadrilateral in which both the pairs of opposite sides are parallel.

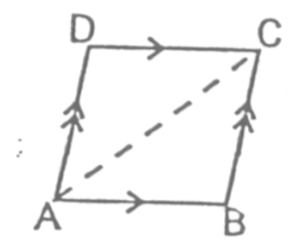
Theorem 1 :

A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Given : A parallelogram ABCD.

To Prove :

A diagonal divides the parallelogram into two congruent triangles



i.e., if diagonal AC is drawn then $\Delta ABC \cong \Delta CDA$

and if diagonal BD is drawn then $\Delta ABD \cong \Delta CDB$

Construction : Join A and C

Proof :

Sine, ABCD is a parallelogram

$AB \parallel DC$ and $AD \parallel BC$

In ΔABC and ΔCDA

$\angle BAC = \angle DCA$ [Alternate angles]

$\angle BCA = \angle DAC$ [Alternate angles]

And, $AC = AC$ [Common side]

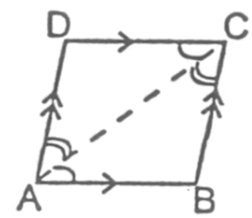
$\therefore \Delta ABC \cong \Delta CDA$ [By ASA]

Similarly, we can prove that

$\Delta ABD \cong \Delta CDB$

Theorem 2 :

In a parallelogram, opposite sides are equal.



Given : A parallelogram ABCD in which $AB \parallel DC$ and $AD \parallel BC$.

To Prove : Opposite sides are equal i.e., $AB = DC$ and $AD = BC$

Construction : Join A and C

Proof :

In ΔABC and ΔCDA

$\angle BAC = \angle DCA$ [Alternate angles]

$\angle BCA = \angle DAC$ [Alternate angles]

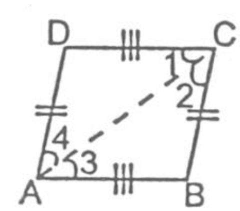
$AC = AC$ [Common]

$\therefore$ $\Delta ABC \cong \Delta CDA$ [By ASA]

$\Rightarrow$ $AB = DC$ and $AD = BC$ [By cpctc] **Hence Proved.**

Theorem 3:

If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram



Given : A quadrilateral ABCD in which

To Prove: ABCD is a parallelogram i.e., $AB \parallel DC$ and $AD \parallel BC$

Construction : Join A and C

Proof :

In ΔABC and ΔCDA

$AB = DC$ [Given]

AD = BC [Given]

And AC = AC [Common]

$\therefore$ $\Delta ABC \cong \Delta CDA$ [By SSS]

$\Rightarrow$ $\angle 1 = \angle 3$ [By cpctc]

And $\angle 2 = \angle 4$ [By cpctc]

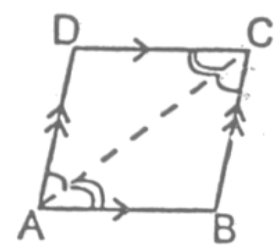
But these are alternate angles and whenever alternate angles are equal, the lines are parallel.

$\therefore$ AB $\parallel$ DC and AD $\parallel$ BC

$\Rightarrow$ ABCD is a parallelogram. **Hence Proved.**

Theorem 4 :

In a parallelogram, opposite angles are equal.



Given : A parallelogram ABCD in which AB $\parallel$ DC and AD $\parallel$ BC.

To Prove : Opposite angles are equal

i.e. $\angle A = \angle C$ and $\angle B = \angle D$

Construction : Draw diagonal AC

Proof : In ΔABC and ΔCDA :

$\angle BAC = \angle DCA$ [Alternate angles]

$\angle BCA = \angle DAC$ [Alternate angles]

AC = AC [Common]

$\therefore \Delta ABC \cong \Delta CDA$ [By ASA]

$\Rightarrow$ $\angle B = \angle D$ [By cpctc]

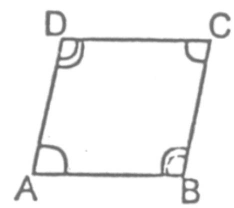
And, $\angle BAD = \angle DCB$ i.e., $\angle A = \angle C$

Similarly, we can prove that $\angle B = \angle D$ **Hence Proved.**

Theorem 5:

If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.

Given : A quadrilateral ABCD in which opposite angles are equal.



i.e., $\angle A = \angle C$ ad $\angle B = \angle D$

To prove : ABCD is a parallelogram i.e, AB ∥ DC and AD ∥ BC.

Proof : Since, the sum of the angles of quadrilateral is 360^0

$\Rightarrow \quad \angle A + \angle B + \angle C + \angle D = 360^0$

$\Rightarrow \quad \angle A + \angle D + \angle A + \angle D = 360..$ $[\angle A = \angle C \text{ and } \angle B = \angle D]$

$\Rightarrow \quad 2\angle A = 2\angle D = 360^0$

$\Rightarrow \quad \angle A + \angle D = 180^0$ [Co-interior angle]

$\Rightarrow \quad AB \parallel DC$

Similarly,

$\angle A + \angle B + \angle C + \angle D = 360^0$

$\Rightarrow \quad \angle A + \angle B + \angle A + \angle B = 360^0$ $[\angle A = \angle C \text{ and } \angle B = \angle D]$

$\Rightarrow \quad 2\angle A + 2\angle V = 360^0$

$\Rightarrow \quad \angle A + \angle B = 180^0$

[∵This is sum of interior angles on the same side of transversal AB]

$\therefore \quad AD \parallel BC$

So, AB ∥ DC and AD ∥ BC

$\Rightarrow$ ABCD is a parallelogram. **Hence Proved.**

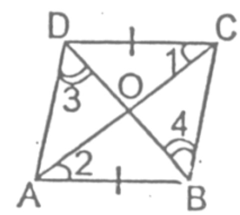
Theorem 6 :

The diagonal of a parallelogram bisect each other.

Given : A parallelogram ABCD. Its diagonals AC and BD intersect each other at point O.

To Prove : Diagonals AC and BD bisect each other i.e., OA = OC and OB = OD.

Proof :



In ΔAOB and ΔCOD

$\because$ AB $\parallel$ DC and BD is a transversal line.

$\therefore \angle ABO = \angle DCO$ [Alternate angles]

$\therefore$ AB $\parallel$ DC and AC is a transversal line.

$\therefore \angle BAO = \angle DCO$ [Alternate angles]

And, AB = DC

$\Rightarrow$ $\Delta AOB \cong \Delta COD$ [By ASA]

$\Rightarrow$ OA = OC and OB = OD [By cpctc] **Hence Proved.**

Theorem 7 :

If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

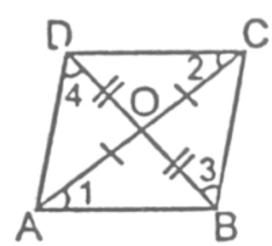
Given : A quadrilateral ABCD whose diagonals AC and BD bisect each other at point O.

i.e., OA = OC and OB = OD

To prove : ABCD is a parallelogram

i.e., AB $\parallel$ DC and AD $\parallel$ BC.

Proof :



In ΔAOB and ΔCOD

OA = OC [Given]

OB = OD [Given}

And, $\angle AOB = \angle COD$ [Vertically opposite angles]

$\Rightarrow \Delta AOB \cong \Delta COD$ [By SAS]

$\Rightarrow \angle 1 = \angle 2$ [By cpctc]

But these are alternate angles and whenever alternate angles are equal, the lines are parallel.

$\therefore$ AB is parallel to DC i.e., AB $\parallel$ DC

Similarly,

$\Delta AOD \cong \Delta COB$ [By SAS]

$\Rightarrow \angle 3 = \angle 4$

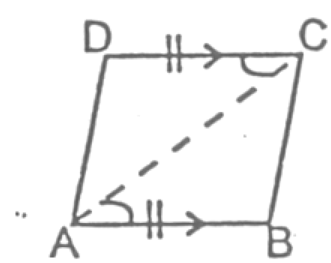
But these are also alternate angles $\Rightarrow$ AD $\parallel$ BC

AB $\parallel$ DC and AD $\parallel$ BC $\Rightarrow$ ABCD is parallelogram. **Hence Proved.**

Theorem 8 :

A quadrilateral is a parallelogram, if a pair of opposite sides is equal and parallel.

Given : A quadrilateral ABCD in which AB $\parallel$ DC and AB = DC.



To Prove : ABCD is a parallelogram

i.e., AB $\parallel$ DC and AD $\parallel$ BC.

Construction : Join A and C.

Proof :

Since AB is parallel to DC and AC is transversal

$\angle BAC = \angle DCA$ [Alternate angles]

AB = DC [Given]

And AC = AC [Common side]

$\Rightarrow$ $\Delta BAC \cong \Delta DCA$ [By SAS]

$\Rightarrow$ $\angle BCA = \angle DAC$ [By cpctc]

But these are alternate angles and whenever alternate angles are equal, the lines are parallel.

$\Rightarrow$ $AD \parallel BC$

Now, $AB \parallel DC$ (given) and $AD \parallel BC$ [Proved above]

$\Rightarrow$ ABCD is a parallelogram **Hence Proved.**

REMARKS :

In order to prove that given quadrilateral is parallelogram, we have to prove that :

(i) Opposite angles of the quadrilateral are equal, or

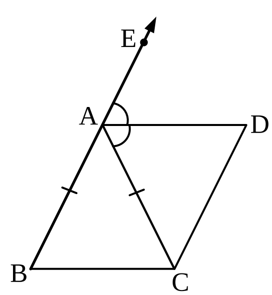
(ii) Diagonals of the quadrilateral bisect each other, or

(iii) A pair of opposite sides is parallel and is of equal length, or

(iv) Opposite sides are equal.

(v) Every diagonal divides the parallelogram into two congruent triangles.

Ex.1 In figure, AB = AC, $\angle EAD = \angle CAD$ and CD || AB. Show that ABCD is a parallelogram.



Sol. In ΔABC, AB = AC [Given]

$\Rightarrow$ $\angle ABC = \angle ACB$ ....(1)

(Angles opposite the equal sides are equal)

$\angle EAD = \angle CAD$ [Given] ....(2)

Now, $\angle EAC = \angle ABC + \angle ACB$ (An exterior angle is equal to sum of two interior opposite angles of a triangle)

$\Rightarrow \quad \angle EAD + \angle XAD = \angle ABC + \angle ACB$

$\Rightarrow \quad \angle CAD + \angle CAD = \angle ACB + \angle ACB$

By (1) and (2)

$\Rightarrow \quad 2\angle CAD = 2\angle ACB$

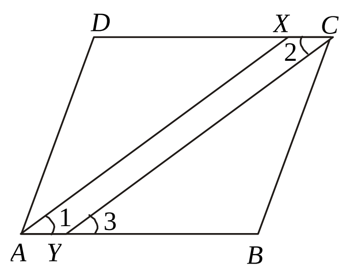
$\Rightarrow \quad \angle CAD = \angle ACB$

$\Rightarrow \quad BC \,||\, AD$

Also, $CD \,||\, AB$ [Given]

Thus, we have both pairs of opposite sides of quadrilateral ABCD parallel. Therefore, ABCD is a parallelogram.

Ex.2 ABCD is a parallelogram and line segments AX,CY are angle bisector of A and C respectively then show AX || CY.



Sol. Since opposite angles are equal in a parallelogram. Therefore, in parallelogram ABCD, we have $\angle A = \angle C$

$\Rightarrow \frac{1}{2}\angle A = \frac{1}{2}\angle C$

$\Rightarrow \angle 1 = \angle 2$ ....(i)

[$\because$ AX and CY are bisectors of $\angle A$ and $\angle C$ respectively]

Now, AB || DC and the transversal CY intersects them.

$\therefore \quad \angle 2 = \angle 3$ ....(ii)

[∵ Alternate interior angles are equal]

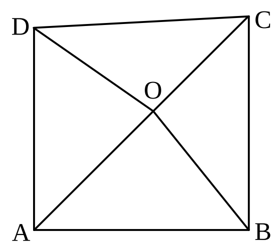
From (i) and (ii), we get

$\angle 1 = \angle 3$

Thus, transversal AB intersects AX and YC at A and Y such that $\angle 1 = \angle 3$ i.e. corresponding angles are equal.

∴ AX || CY

Ex.3 In the adjoining figure, a point O is taken inside an equilateral quad. ABCD such that OB = OD. Show that A, O and C are in the same straight line.



Sol. Given a quad. ABCD in which AB = BC = CD = DA and O is a point within it such that OB = OD.

To prove $\angle AOB + \angle COB = 180º$

Proof In ΔOAB and OAD, we have

AB = AD (given), OA = OA

(common) and OB = OD (given)

∴ $\Delta OAB \cong \Delta OAD$

∴ $\angle AOB = \angle AOD$ ....(i) (c.p.c.t.)

Similarly, $\Delta OBC \cong \Delta ODC$

∴ $\angle COB = \angle COD$ ....(ii)

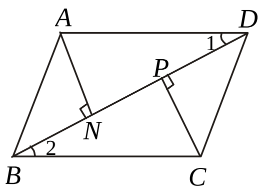
Now, $\angle AOB + \angle COB + \angle COD + \angle AOD = 360º$ [∠ at a point]

$\Rightarrow 2(\angle AOB + \angle COB) = 360º$

$\Rightarrow \angle AOB + \angle COB = 180º$

Ex.4 In figure AN and CP are perpendiculars to the diagonal BD of a parallelogram ABCD. Prove that :

(i) $\Delta ADN \cong \Delta CBP$ (ii) $AN = CP$



Sol. Since ABCD is a parallelogram.

$\therefore$ AD || BC

Now, AD || BC and transversal BD intersects them at B and D.

$\therefore \angle 1 = \angle 2$

[$\because$ Alternate interior angles are equal]

Now, in Δs ADN and CBP, we have

$\angle 1 = \angle 2$

$\angle AND = \angle CPD$ and, AD = BC

[$\because$ Opposite sides of a $||^{gm}$ are equal]

So, by AAS criterion of congruence

$\Delta ADN \cong \Delta CBP$

AN = CP

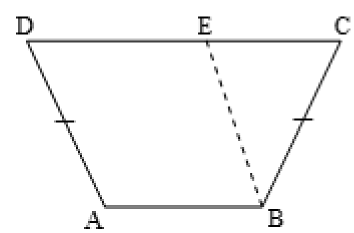
[$\because$ Corresponding parts of congruent triangles are equal]

Ex.5 In figure, ABCD is a trapezium such that AB || CD and AD = BC.

BE || AD and BE meets BC at E. Show that

(i) ABED is a parallelogram.

(ii) $\angle A + \angle C = \angle B + \angle D = 180º$.



Sol. Here, AB || CD (Given)

$\Rightarrow$ AB || DE ....(1)

Also, BE || AD (Given) ....(2)

From (1) and (2),

ABED is a parallelogram

$\Rightarrow$ AD = BE ....(3)

Also, AD = BC (Given) ....(4)

From (3) and (4),

BE = BC

$\Rightarrow$ $\angle BEC = \angle BCE$ ....(5)

Also, $\angle BAD = \angle BED$

(opposite angles of parallelogram ABED)

i.e., $\angle BED = \angle BAD$ ....(6)

Now, $\angle BED + \angle BEC = 180º$ (Linear pair of angles)

$\Rightarrow$ $\angle BAD + \angle BCE = 180º$

By (5) and (6)

$\Rightarrow$ $\angle A + \angle C = 180º$

Similarly, $\angle B + \angle D = 180º$