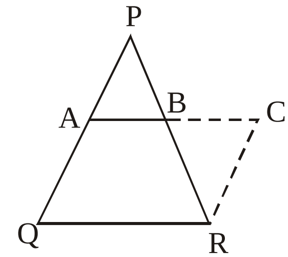
QUADRILATERALS

MID POINT THEOREM

THEOREM : 10 (MID-POINT THEOREM) :

In a triangle, the line segment joining the mid-points of any two sides is parallel to the third side and also half of it.



Given : In ΔPQR, A and B are the mid-points of $\overline{PQ}$ and $\overline{PR}$ respectively.

To Prove : $\overline{AB} \| \overline{QR}$ and $AB = \frac{1}{2} QR$

Construction : Draw $\overline{RC}$ parallel to $\overline{QA}$ to meet produced $\overline{AB}$ at C.

Proof :

(i) In ΔABP and ΔCBR,

$\angle PBA = \angle RBC$ (vertically opposite angles)

$\angle PAB = \angle RCB$ (alternate angles and $\overline{CR} \| \overline{PQ}$)

$PB = BR$ (B is mid point of PR)

By AAS congruence property,

$\Delta ABP \cong \Delta CBR$

$\therefore$ $PA = CR$ and $AB = BC$ (corresponding parts of congruent triangles)

$\Rightarrow$ $AQ = CR$ ($\because PA = AQ$)

In quadrilateral ACRQ, $AQ = CR$ and $\overline{AQ} \| \overline{CR}$

$\therefore$ ACRQ is a parallelogram.

$\therefore$ $\overline{AC} \| \overline{QR} \Rightarrow \overline{AB} \| \overline{QR}$

(ii) $AC = QR$ (opposite sides of parallelogram)

$\Rightarrow$ $QR = AB + BC$

$\Rightarrow$ $QR = 2AB$ ($\because AB = BC$)

$\Rightarrow$ $AB = \frac{1}{2} QR$

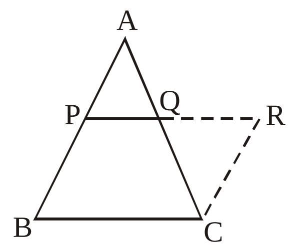
THEOREM : 11 (CONVERSE OF MID POINT THEOREM) :

The line drawn through the mid-point of one side of a triangle parallel to the another side, bisects the third side.

Given : A triangle ABC in which P is the mid-point of side AB and PQ is parallel to BC.

To prove : PQ bisects the third side AC i.e., AQ = QC.

Construction : Through C, draw CR parallel to BA, which meets PQ produced at point R.



Proof :

Since, PQ || BC i.e., PR || BC [Given]

and CR || BA i.e., CR || BP [By construction]

$\therefore$ Opposite side of quadrilateral PBCR are parallel.

$\Rightarrow$ PBCR is a parallelogram.

$\Rightarrow$ BP = CR

Also, AP = PB [As, P is mid-point of AB]

$\therefore$ CR = AP

AB || CR and AC is transversal,

$\Rightarrow$ $\angle PAQ = \angle RCQ$ (alternate interior angles)

AB || CR and PR is transversal,

$\Rightarrow$ $\angle APQ = \angle CRQ$ (alternate interior angles)

In ΔAPQ and ΔCRQ,

AP = CR, $\angle PAQ = \angle RCQ$ and $\angle APQ = \angle CRQ$

$\Rightarrow$ $\Delta APQ \cong \Delta CRQ$ (By ASA congruency)

$\Rightarrow$ AQ = CQ (CPCT)

Ex.1 Prove that the four triangles formed by joining in pairs, the mid-points of three sides of a triangle, are concurrent to each other.

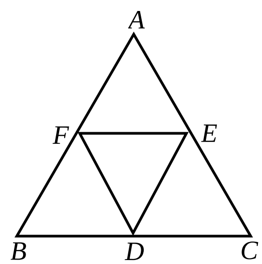
Sol. **Given :** A triangle ABC and D,E,F are the mid-points of sides BC, CA and AB respectively.

To Prove : Δ AFE $\cong$ ΔFBD $\cong$ ΔEDC $\cong$ ΔDEF.

Proof :

Since the segment joining the mid-points of the sides of a triangle is half of the third side.

Therefore,



$DE = \frac{1}{2}AB \Rightarrow \quad DE = AF = BF \quad .... (i)$

$EF = \frac{1}{2}BC \Rightarrow \quad EF = BD = CD \quad .... (ii)$

$DF = \frac{1}{2}AC \Rightarrow \quad DF = AE = EC \quad ....(iii)$

Now, in Δs DEF and AFE, we have

DE = AF [From (i)]

DF = AE [From (ii)]

and, EF = FE [Common]

So, by SSS criterion of congruence,

Δ DEF $\cong$ Δ AFE

Similarly, Δ DEF $\cong$ Δ FBD and Δ DEF $\cong$ Δ EDC

Hence, Δ AFE $\cong$ Δ FBD $\cong$ Δ EDC $\cong$ Δ DEF

Ex.2 Let ABC be an isosceles triangle with AB = AC and let D,E,F be the mid-points of BC, CA and AB respectively. Show that AD $\perp$ FE and AD is bisected by FE.

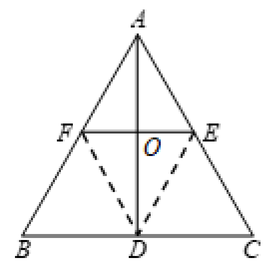
Sol. **Given :** An isosceles triangle ABC with D, E and F as the mid-points of sides BC, CA and AB respectively such that AB = AC. AD intersects FE at O.

To Prove : AD $\perp$ FE and AD is bisected by FE.

Constructon : Join DE and DF.

Proof :

Since the segment joining the mid-points of two sides of a triangle is parallel to third side and is half of it. Therefore,



DE || AB and DE = $\frac{1}{2}$AB

Also, DF || AC and DF = $\frac{1}{2}$AC

But, AB = AC [Given]

$\Rightarrow$ $\frac{1}{2}$AB = $\frac{1}{2}$AC

$\Rightarrow$ DE = DF .... (i)

Now, DE = $\frac{1}{2}$AB $\Rightarrow$ DE = AF .... (ii)

and, DF = $\frac{1}{2}$AC $\Rightarrow$ DF = AE ....(iii)

From (i), (ii) and (iii) we have

DE = AE = AF = DF

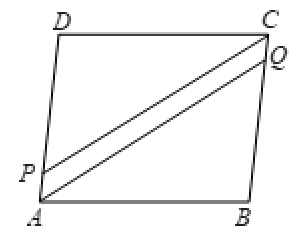
$\Rightarrow$ DEAF is a rhombus.

$\Rightarrow$ Diagonals AD and FE bisect each other at right angle.

AD $\perp$ FE and AD is bisected by FE.

Ex.3 ABCD is a parallelogram. P is a point on AD such that AP = $\frac{1}{3}$ AD and Q is a point on BC such that CQ = $\frac{1}{3}$BP. Prove that AQCP is a parallelogram.

Sol. ABCD is a parallelogram.



$\Rightarrow$ AD = BC and AD || BC

$\Rightarrow \frac{1}{3}AD = \frac{1}{3}BC$ and AD || BC

$\Rightarrow$ AP = CQ and AP || CQ

Thus, APCQ is a quadrilateral such that one pair of opposite side AP and CQ are parallel and equal.

Hence, APCQ is a parallelogram.