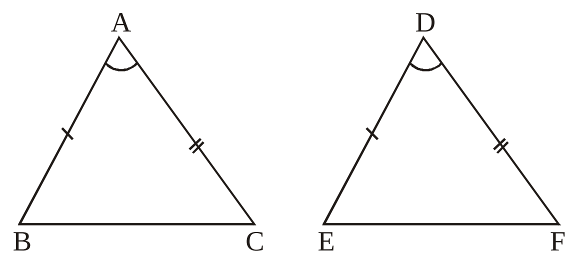
TRIANGLES

CRITERIA OF CONGRUENCE OF TRIANGLE

CRITERIA FOR CONGRUENCE OF TRIANGLES :

1 Side-Angle-Side (SAS) Rule for congruency of two triangles :

Statement : If two triangles have two sides and the included angle of the one triangle equal to the corresponding sides and the included angle of the other triangle, then the triangles are congruent.



Given : ΔABC and ΔDEF in which $AB = DE$, $AC = DF$ and $\angle A = \angle D$

To Prove : $\Delta ABC \cong \Delta DEF$

Proof : Place ΔABC over ΔDEF such that A falls on D and AB falls along DE.

Since $AB = DE$, so B falls on E.

Since, $\angle A = \angle D$, so AC will fall along DF.

Also, $AC = DF$

$\therefore$ C will fall on F.

Thus, AC will coincide with DF.

And, therefore, BC will coincide with EF.

$\therefore$ ΔABC coincides with ΔDEF.

Hence, $\Delta ABC \cong \Delta DEF$

NOTE : It is very important that the equal angles are included between the pair of equal sides. So, SAS congruence rule holds but not ASS or SSA rule.

2 Angle-Side-Angle (ASA) Rule for congruency of two triangles :

Statement : Two triangles are congruent if any two angles and the included side of one triangle are equal to the two angles and the included side of the other triangle.

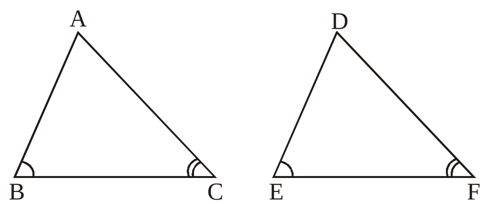
Given : ΔABC and ΔDEF

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$

To Prove : $\Delta ABC \cong \Delta DEF$

Proof : On comparing the sides AB and DE of ΔABC and ΔDEF, there are three possibilities:

(i) $AB = DE$ (ii) $AB < DE$ and (iii) $AB > DE$



Case (i) : When $AB = DE$; then in ΔABC and ΔDEF

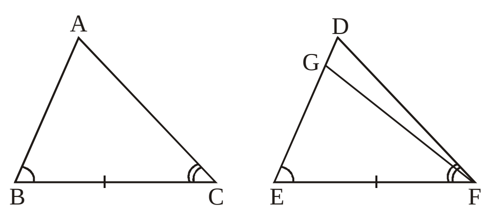
$AB = DE$ (Assumed)

$\angle ABC = \angle DEF$ (Given)

$BC = EF$ (Given)

Hence, $\Delta ABC \cong \Delta DEF$ (By SAS criterian)

Case (ii) : When $AB < DE$, then we take point G on DE



such that $AB = GE$ and join GF as shown in figure.

Now in ΔABC and ΔDEF

$AB = GE$ (Assumed)

$BC = EF$ (Given)

$\angle ABC = \angle GEF$ [$\angle GEF = \angle DEF$]

$\Delta ABC \cong \Delta GEF$ (By SAS Rule)

Hence, $\angle ACB = \angle GFE$ .....(1)

But, $\angle ACB = \angle DFE$ .....(2)

Therefore from (1) and (2), we have

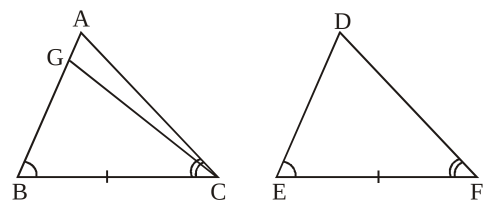
$\angle DFE = \angle GFE$

which is impossible unless GF coincide with DF and hence G coincides with D.

$\therefore$ $AB = DE$

Hence, $\Delta ABC \cong \Delta DEF$ (By SAS rule)

Case (iii) : When $AB > DE$, then we take a point G on AB such that BG =DE and show as in case (ii) that G must coincide with A.

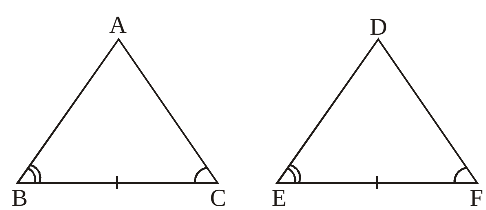


i.e. $AB = DE$ and hence, $\Delta ABC \cong \Delta DEF$ (SAS rule)

Since the sum of the three interior angles of a triangle is 180°, therefore if two angles of one triangle are equal to the two angles of another triangle, then the third angle of the first will automatically be equal to the third angle of the second triangle.

Angle-Angle-Side (AAS) Congruence Rule :

Statement : Two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.



Given : In ΔABC and ΔDEF,

$\angle A = \angle D$, $\angle B = \angle E$ and $BC = EF$

To Prove : $\Delta ABC \cong \Delta DEF$

Proof : The sum of three interior angles of a triangle is 180°, then

$\angle A + \angle B + \angle C = 180°$ .....(1)

$\angle D + \angle E + \angle F = 180°$ .....(2)

From (1) and (2)

$\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$ .....(3)

$\angle B = \angle E$

$\angle A = \angle D$ (Given)

Hence, $\angle C = \angle F$ ....(4) [By (3)]

Now, ΔABC and ΔDEF

$\angle B = \angle E$ (Given)

$BC = EF$ (Given)

$\angle C = \angle F$ [By (4)]

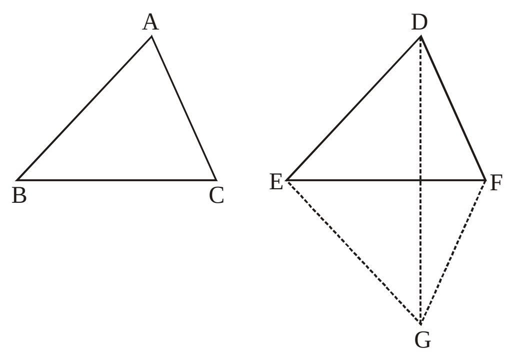
Hence, $\Delta ABC \cong \Delta DEF$ (By ASA rule)

Side-Side-Side (SSS) Rule for congruency of two triangles :

Statement : Two triangles are congruent if the three sides of one triangle are equal to the three sides of the other triangle.

Given : ΔABC and ΔDEF in which $AB = DE$, $BC = EF$ and $AC = DF$

To Prove : $\Delta ABC \cong \Delta DEF$



Construction: Draw a line segment EG at the other side of ED with respect to EF such that $EG = AB$ and $\angle ABC = \angle FEG$ then join GF and DG.

Proof : In ΔABC and ΔGEF

$AB = GE$ (By construction)

$\angle ABC = \angle GEF$ (By construction)

$BC = EF$

Hence by SAS, rule

$\Delta ABC \cong \Delta GEF$

Hence the corresponding sides and angles are equal

$\therefore \quad \angle A = \angle EGF, AC = GF \quad$ .....(1)

Now $\quad AB = EG \quad$ (By construction)

$AB = DE \quad$ (Given)

$GE = DE \quad$ .....(2)

Similarly

$AC = GF \quad$ [From (1)]

and $\quad AC = DF \quad$ (Given)

$\therefore \quad GF = DF \quad$ .....(3)

Now in ΔEDG the opposite angle of equal sides EG and DE are equal

$\therefore \quad \angle EDG = \angle EGD \quad$ .....(4)

Similarly in ΔFDG the opposite angle of equal sides GF and DF are equal

$\angle GDF = \angle DGF \quad$ .....(5)

Adding equation (4) and (5), we get

$\angle EDG + \angle GDF = \angle EGD + \angle DGF$

$\Rightarrow \quad \angle EDF = \angle EGF \quad . \quad$ .....(6)

From equation (1),

$\angle A = \angle EGF \quad$ .....(7)

Hence, from (6) and (7), we have

$\angle A = \angle EDF \quad$ .....(8)

In ΔABC and ΔDEF

$AB = DE \quad$ (Given)

$\angle A = \angle EDF \quad$ [By (8)]

$AC = DF$

By SAS rule, $\Delta ABC \cong \Delta DEF$

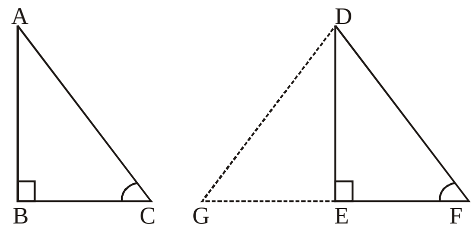
Right-Angle-Hypotenuse-Side (RHS) Rule for congruency of two triangles:

Statement : Two right triangles are congruent if the hypotenuse and a side of one triangle are equal to hypotenuse and a side of the other triangle respectively.

Given : Two right triangles ΔABC and ΔDEF, in which $\angle B = \angle E = 90°$,

Hypotenuse AC = Hypotenuse DF and Side AB = Side DE

To Prove : $\Delta ABC \cong \Delta DEF$



Construction : Produce FE to G such that $GE = BC$ and join G and D.

Proof : In ΔDEF,

$\angle DEF = 90°$

$\therefore \quad \angle DEG = 90°$ .....(1)

Now in ΔABC and ΔDEG

$AB = DE$ (Given)

$BC = GE$ (By construction)

$\angle ABC = \angle DEG = 90°$ [By(1)]

Therefore, by SAS rule, $\Delta ABC \cong \Delta DEG$

Hence the corresponding sides and angles are equal

$\therefore \quad AC = DG$ and $\angle C = \angle G$ .....(2)

But given that $AC = DF$ .....(3)

From (2) and (3)

$DG = DF$ .....(4)

(Angles opposite to the equal sides DG and DF of ΔDGF are equal)

Hence $\angle G = \angle F$ .....(5)

Therefore, from (2) and (5)

$\angle C = \angle F$ .....(6)

Now in ΔABC and ΔDEF,

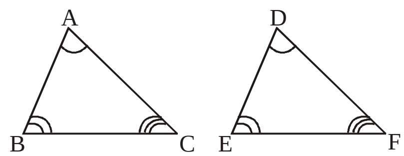
$\angle C = \angle F$ [From (6)]

$\angle ABC = \angle DEF = 90°$ (Given)

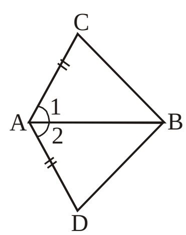
and $AB = DE$

Hence by AAS rule, $\Delta ABC \cong \Delta DEF$

If all the three angles of one triangle are equal to all the three angles of another triangle. Then the two triangles need not to be congruent.



Ex.1 In fig. ACBD is a quadrilateral in which AC=AD and AB bisects $\angle A$. Show that $\Delta ABC \cong \Delta ABD$ and $BC = BD$.



Sol: In ΔABC and ΔABD,

$AC = AD$ [Given]

$AB = AB$ [Common]

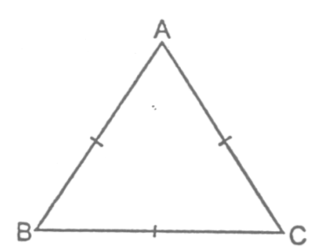
$\angle 1 = \angle 2$ [AB is the bisector of $\angle A$]

Thus $\Delta ABC \cong \Delta ABD$ [SAS rule of congruence]

Therefore, $BC = BD$ [By CPCT]

Ex.2 Prove that measure of each angle of an equilateral triangle is 60^0.

Sol. Let ΔABC be an equilateral triangle, then we have



$AB = BC = CA$ ...(i)

$\therefore$ $AB = BC$

$\therefore$ $\angle C = \angle A$ ...(ii) [Angles opposite to equal sides are equal]

Also, $BC = CA$

$\therefore \quad \angle A = \angle B$ ...(iii) [Angles opposite to equal sides]

By (ii) & (iii) we get $\angle A = \angle B = \angle C$

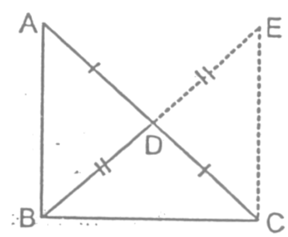
Now in ΔABC $\angle A + \angle B + \angle C = 180^0$

$\Rightarrow \quad 3\angle A = 180^0$ [$\therefore \angle A = \angle B = \angle C$]

$\Rightarrow \quad \angle A = 60^0 = \angle B = \angle C$ **Hence Proved.**

Ex.3 If D is the mid-point of the hypotenuse AC of a right triangle ABC, prove that BD = $\frac{1}{2}$ AC.

Sol. Let ΔABC is a right triangle such that $\angle B = 90^0$ and D is mid point of AC then we have to prove that BD = $\frac{1}{2}$AC we produce BD to E such that BD = AC and EC.



Now is ΔADB and ΔCDE we have

AD = DC [Given]

BD = DE [By construction]

And, $\angle ADB = \angle CDE$ [Vertically opposite angles]

$\therefore$ By SAS criterion of congruence we have

$\Delta ADB \cong \Delta CDE$

$\Rightarrow \quad EC = AB$ and $\angle CED = \angle ABD$ ....(i) [By cpctc]

But $\angle CED$ & $\angle ABD$ are alternate interior angles

$CE \parallel AB \Rightarrow \angle ABC + \angle ECB = 180^0$ [Consecutive interior angles]

$\Rightarrow \quad 90 + \angle ECB = 180^0$

$\Rightarrow \quad \angle ECB = 90^0$

Now, In ΔABC & ΔECB we have

$AB = EC$ [By (i)]

$BC = BC$ [Common]

And, $\angle ABC = \angle ECB = 90^0$

$\therefore$ BY SAS criterion of congruence

$\Delta ABC \cong \Delta ECB$

$\Rightarrow AC = EB$ [By cpctc]

$\Rightarrow \frac{1}{2}AC = \frac{1}{2}EB$

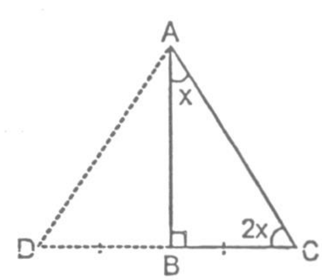
$\Rightarrow BD = \frac{1}{2}AC$ **Hence Proved.**

Ex.4 In a right angled triangle, one acute angle is double the other. Prove that the hypotenuse is double the smallest side.

Sol. Let ΔABC is a right triangle such that $\angle B = 90^0$ and $\angle ACB = 2\angle CAB$, then we have to prove $AC = 2BC$.

we produce CB to D such that $BD = CB$ and join AD.

Proof : In ΔABD and ΔABC we have



$BD = BC$ [By construction]

$AB = AB$ [Common]

$\angle ABD = \angle ABC = 90^0$

$\therefore$ By SAS criterion of congruence we get

$\Delta ABD \cong \Delta ABC$

$\Rightarrow AD = AC$ and $\angle DAB = \angle CAB$ [By cpctc]

$\Rightarrow AD = AC$ and $\angle DAB = x$ [$\therefore \angle CAB = x$]

Now, $\angle DAC = \angle DAB + \angle CAB = x + x = 2x$

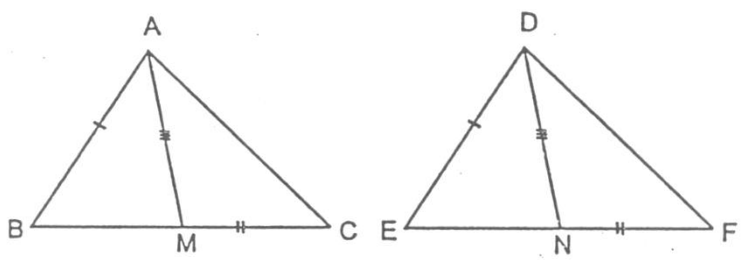
$\therefore$ $\angle DAC = \angle ACD$

$\Rightarrow$ $DC = AD$ [Side Opposite to equal angles]

$\Rightarrow$ $2BC = AD$ [$\because DC = 2BC$

$\Rightarrow$ $2BC = AC$ [$AD = AC$] **Hence Proved.**

Ex.5 In figure, two sides AB and BC and the median AM of a ΔABC are respectively equal to sides DE and EF and the median DN of ΔDEF. Prove that $\Delta ABC \cong \Delta DEF$.



Sol. $\therefore$ AM and DN are medians of ΔABC & ΔDEF respectively

$\therefore$ $BM = MC$ & $EN = NF$

$\Rightarrow$ $BM = \frac{1}{2}BC$ & $EN = \frac{1}{2}EF$

But, $BC = EF$ $\therefore BM = EN$ ...(i)

In ΔABM & ΔDEN we have

$AB = DE$ [Given]

$AM = DN$ [Given]

$BM = EN$ [By (i)]

$\therefore$ By SSS criterion of congruence we have

$\Delta ABM \cong \Delta DEN \Rightarrow \angle B = \angle E$ ...(ii) [By cpctc]

Now, In ΔABC & ΔDEF

$AB = DE$ [Given]

$\angle B = \angle E$ [By (ii)]

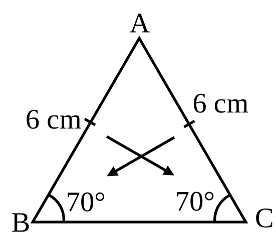
$BC = EF$ [Given]

$\therefore$ By SAS criterion of congruence we get

$\Delta ABC \cong \Delta DEF$

ISOSCELES TRIANGLE :

A triangle in which two sides are equal & opposite angles of these two lines are also equal.



$AB = AC = 6$ cm, $\angle B = \angle C = 70°$