POLYNOMIALS

FACTORISATION

FACTORIZATION OF POLYNOMIALS :

Factors:

A polynomial g(x) is called a factor of the polynomial p(x) if g(x) divides p(x) exactly.

For example, $(x-2)$ is a factor of $(x^2 + 3x - 10)$

Factorization :

Factorization is expressing a given polynomial as a product of two or more polynomials.

For example : $(x^2 - 16) = (x - 4)(x + 4)$; $(x^2 - 3x + 2) = (x - 2)(x - 1)$

Methods of Factorization :

(i) By taking out the common factor :

When each term of an expression has a common factor, we divide each term by this factor and take it out as a multiple.

Ex.1: Factorize : (i) $5x^2 - 20xy$ (ii) $6(2a + 3b)^2 - 8(2a+3b)$

Sol: (i) $5x^2 - 20xy = 5x(x - 4y)$

(ii) $6(2a + 3b)^2 - 8(2a + 3b) = 2(2a + 3b)[3(2a + 3b) - 4]$

$= 2(2a + 3b)[6a + 9b - 4]$

(ii) Factorization the Difference of Two Squares

Difference of two squares can be factorized by using the algebraic identity

$(x^2 - y^2) = (x + y)(x - y)$

Ex.2: Factorize : (i) $9x^2 - 16y^2$ (ii) $x^3 - x$

Sol: (i) $(3x)^2 - (4y)^2 = (3x + 4y)(3x - 4y)$

(ii) $x(x^2 - 1) = x(x + 1)(x - 1)$

Ex.3 Factorise : $4(2a + 3b - 4c)^2 - (a - 4b + 5c).2$

Sol. $= 4(2a + 3b - 4c)^2 - (a - 4b + 5c)^2$

$= [2(2a + 3b - 4c)]^2 - (a - 4b + 5c)^2$

$= [4a + 6b - 8c + a - 4b + 5c]\ [4a + 6b - 8c - a + 4b - 5c]$

$= [5a + 2b - 3c]\ [3a + 10b - 13c]$

Ex.4 Factorise : $4x^2 + \frac{1}{4x^2} + 2 - 9y^2$.

Sol. $4x^2 + \frac{1}{4x^2} + 2 - 9y^2$

$= (2x)^2 + 2\,(2x).\left(\frac{1}{2x}\right) + \left(\frac{1}{2x}\right)^2 - (3y)^2$

$= \left(2x + \frac{1}{2x}\right)^2 - (3y)^2$

$= \left(2x + \frac{1}{2x} + 3y\right)\left(2x + \frac{1}{2x} - 3y\right)$

(iii) Factorizing of Quadratic Trinomials

Spliting the middle term method :

For polynomials of the form $ax^2 + bx + c$, we find integers p and q such that $p + q = b$ and $pq = ac$. Then,

$ax^2 + bx + c = ax^2 + (p+q)x + pq/a$

$= [a^2x^2 + apx + aqx + pq] / a$

$= [ax (ax + p) + q (ax + p)] / a$

$= [(ax + p) (ax +q)] /a$

Hence, $(ax^2 + bx + c) = [(ax + p) (ax + q)] / a$

Ex.5: Factorize : $6x^2 + 7x - 3$

Sol: The given expression is $6x^2 + 7x - 3$. Here $6 \times (-3) = -18$

So, we try to split 7 into two parts whose sum is 7 and product −18.

Clearly, $9 + (-2) = 7$ and $9 \times (-2) = -18$

$$6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3$$
$$= 3x(2x + 3) - 1(2x + 3)$$
$$= (2x + 3)(3x - 1)$$

Hence, $6x^2 + 7x - 3 = (2x + 3)(3x - 1)$

Ex.6 : Factorize : $2x^2 - \frac{5x}{6} + \frac{1}{12}$

Sol: We have

$$= \frac{(24x^2 - 10x + 1)}{12} \quad = \frac{1}{12}(24x^2 - 10x + 1)$$

$$= \frac{1}{12}(24x^2 - 6x - 4x + 1)$$

$$= \frac{1}{12}[6x(4x - 1) - 1(4x - 1)]$$

$$= \frac{1}{12}(4x - 1)(6x - 1)$$

$$2x^2 - \frac{5x}{6} + \frac{1}{2} = \frac{1}{12}(4x - 1)(6x - 1)$$

Ex.7 $81a^2b^2c^2 + 64a^6b^2 - 144a^4b^2c$

Sol. $81a^2b^2bc^2 + 64a^6b^2 - 144a^4b^2c$

$= [9abc]^2 - 2[9abc][8a^3b] + [8a^3b]^2$

$= [9abc - 8a^3b]^2 = a^2b^2[9c - 8a^2]^2$

(iv) Factorization of Sum or Difference of Cubes

The sum and difference of cubes is factorized by using given algebraic identities

(a) $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$

(b) $(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$

Ex.8: Factorize : $27x^3 + 125y^3$

Sol: Using the identity $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$, we have

$(27x^3 + 125y^3) = (3x)^3 + (5y)^3$

$= (3x + 5y)[(3x)^2 - (3x)(5y) + (5y)^2]$

$= (3x + 5y)(9x^2 - 15xy + 25y^2)$

Ex.9: Factorize : $a^3 - b^3 - a + b$

Sol: $a^3 - b^3 - a + b$

$= (a^3 - b^3) - (a - b)$

$= (a - b)(a^2 + ab + b^2) - (a - b)$ [Since, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

$= (a - b)(a^2 + ab + b^2 - 1)$

Hence, $(a^3 - b^3 - a + b) = (a - b)(a^2 + ab + b^2 - 1)$

Ex.10 Factorise : $64a^{13}b + 343ab^{13}$.

Sol. $64a^{13}b + 343ab^{13} = ab[64a^{12} + 343b^{12}]$

$= ab[(4a^4)^3 + (7b^4)^3]$

$= ab[4a^4 + 7b^4][(4a^4)^2 - (4a^4)(7b^4) + (7b^4)^2]$

$= ab[4a^4 + 7b^4][16a^8 - 28a^4b^4 + 49b^8]$

ALGEBRAIC IDENTITY

An identity is an equality which is true for all values of the variables

Some important identities are

(i) $(a + b)^2 = a^2 + 2ab + b^2$

(ii) $(a - b)^2 = a^2 - 2ab + b^2$

(iii) $a^2 - b^2 = (a + b)(a - b)$

(iv) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(v) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(vi) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(vii) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(viii) $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$

(ix) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Special case : if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$.

(a) Value Form :

(i) $a^2 + b^2 = (a + b)^2 - 2ab$, if $a + b$ and ab are given

(ii) $a^2 + b^2 = (a - b)^2 + 2ab$ if $a - b$ and ab are given

(iii) $a + b = \sqrt{(a-b)^2+4ab}$ if $a - b$ and ab are given

(iv) $a - b = \sqrt{(a+b)^2-4ab}$ if $a + b$ and ab are given

(v) $a^2 + \frac{1}{a^2} = \left(a+\frac{1}{a}\right)^2 - 2$ if $a + \frac{1}{a}$ is given

(vi) $a^2 + \frac{1}{a^2} = \left(a+\frac{1}{a}\right)^2 + 2$ if $a - \frac{1}{a}$ is given

(vii) $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ if $(a + b)$ and ab are given

(viii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$ if $(a - b)$ and ab are given

(ix) $x^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right)$ if $a + \frac{1}{a}$ is given

(x) $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 + 3\left(a - 1\frac{1}{a}\right)$, if $\left(a - \frac{1}{a}\right)$ is given

(xi) $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = [(a + b)^2 - 2ab]^2 - 2a^2b^2$, if $(a + b)$ and ab are given

(xii) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = [(a + b)^2 - 2ab](a + b)(a - b)$

(xiii) $a^5 + b^5 = (a^3 + b^3)(a^2 + b^2) - a^2b^2(a + b)$

Ex.11 Find the value of :

(i) $36x^2 + 49y^2 + 84xy$, when $x = 3, y = 6$

(ii) $25x^2 + 16y^2 - 40xy$, when $x = 6, y = 7$

Sol. (i) $36x^2 + 49y^2 + 84xy = (6x)^2 + (7y)^2 + 2 \times (6x) \times (7y)$

$= (6x + 7y)^2$

$= (6 \times 3 + 7 \times 6)^2$ [When $x = 3, y = 6$]

$= (18 + 42)^2$

$= (60)^2$

$= 3600.$

(ii) $25x^2 + 16y^2 - 40xy = (5x)^2 + (4y)^2 - 2 \times (5x) \times (4y)$

$= (5x - 4y)^2$

$= (5 \times 6 - 4 \times 7)^2$ [When $x = 6, y = 7$]

$= (30 - 28)^2 = 2^2 = 4$

Ex.12 If $x^2 + \frac{1}{x^2} = 23$, find the value of $\left(x+\frac{1}{x}\right)$.

Sol. $x^2 + \frac{1}{x^2} = 23$ ....(i)

$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 25$ [Adding 2 on both sides of (i)]

$\Rightarrow (x^2) + \left(\frac{1}{x}\right)^2 + 2 . x . \frac{1}{x} = 25$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 = (5)^2$

$\Rightarrow x + \frac{1}{x} = 5$

Ex.13 Prove that $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$.

Sol. Here, L.H.S. $= a^2 + b^2 + c^2 - ab -+ bc - ca$

$= \frac{1}{2} [2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$

$= \frac{1}{2}[(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)]$

$= \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

= RHS Hence Proved.

Ex.14 Evaluate :

(i) $(107)^2$ (ii) $(94)^2$ (iii) $(0.99)^2$

Sol. (i) $(107)^2 = (100 + 7)^2$

$= (100)^2 + (7)^2 + 2 \times 100 \times 7$

$= 10000 + 49 + 1400 = 11449$

(ii) $(94)^2 = (100 - 6)^2$

$= (100)^2 + (6)^2 - 2 \times 100 \times 6$

$= 10000 + 36 - 1200$

$= 8836$

(iii) $(0.99)^2 = (1 - 0.01)^2$

$= (1)^2 + (0.01)^2 - 2 \times 1 \times 0.01$

$= \ + 0.0001 - 0.02$

$= 0.9801$

NOTE : We may extend the formula for squaring a binomial to the squaring of a trinomial as given below.

$(a + b + c)^2 = [a + (b + c)]^2$

$= a^2 + (b + c)^2 + 2 \times a \times (b + c)$ [Using the identity for the square of binomial]

$= a^2 + b^2 + c^2 + 2bc + 2 (b + c)$ [Using $(b + c)^2 = b^2 + c^2 + 2bc$]

$= a^2 + b^2 + c^2 + 2bc + 2ab + 2ac$ [Using the distributive law]

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

$\therefore \ (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

Ex.15 Simplify : $(3x + 4)^3 - (3x - 4)^3$.

Sol. We have,

$(3x + 4)^3 - (3x - 4)^3 = [(3x)^3 + (4)^3 + 3 \times 3x \times 4 \times (3x + 4)] - [(3x)^3 - (4)^3 - 3 \times 3x \times 4 \times (3x - 4)]$

$= [27^3 + 64 + 36x (3x + 4)] - [27^3 - 64 - 36x (3x - 4)]$

$= [27x^3 + 64 + 108x^2 + 144x] - [27x^3 - 64 - 108x^2 + 144x]$

$= 27x^3 + 64 + 108x^2 + 144x - 27x^3 + 64 + 108x^2 - 144x$

$= 128 + 216x^2$

$\because\ (3x + 4)^3 - (3x - 4)^3 = 128 + 216x^2$

Ex.16 Evaluate :

(i) $(1005)^3$ (ii) $(997)^3$

Sol. (i) $(1005)^3 = (1000 + 5)^3$

$= (1000)^3 + (5)^3 + 3 \times 1000 \times 5 \times (1000 + 5)$

$= 1000000000 + 125 + 15000 + (1000 + 5)$

$= 1000000000 + 125 + 15000000 + 75000$

$= 1015075125.$

(ii) $(997)^3 = (1000 - 3)^3$

$= (1000)^3 - (3)^3 - 3 \times 1000 \times 3 \times (1000 - 3)$

$= 1000000000 - 27 - 9000 \times (1000 - 3)$

$= 1000000000 - 27 - 900000 + 27000$

$= 991026973$

Ex.17 If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$

Sol. We have, $x-\frac{1}{x}=5$ ...(i)

$\Rightarrow \left(x-\frac{1}{x}\right)^3=(5)^3$ [Cubing both sides of (i)]

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3x.\frac{1}{x}.\left(x - \frac{1}{x}\right) = 125$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 125$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3 \times 5 = 125$ [Substituting $\left(x - \frac{1}{x}\right) = 5$]

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 15 = 125$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = (125 + 15) = 140$