POLYNOMIALS

FACTOR THEOREM

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Let p(x) be a polynomial of degree n ^{3} 1 and let a be any real number.

(i) If p(a) = 0, then (x –a) is a factor of p(x).

(ii) If (x – a) is a factor of p(x) then p(a) = 0

Proof :

By the Remainder Theorem, p(x) = (x – a) q(x) + p(a).

(i) If p(a) = 0, then p(x) = (x – a) q(x), which shows that x – a is a factor of p(x).

(ii) Since x – a is a factor of p(x), p(x) = (x – a) g(x) for same polynomial g(x). In this case,

p(a) = (a – a) g(a) = 0.

NOTE :

(i) If ax – b is a factor of p(x) then $P(\frac{b}{a}) = 0$

(ii) If ax + b is a factor of p(x) then $p(\frac{-b}{a}) = 0$

(iii) (x – a) is a factor of $(x^n - a^n)$ where "n" is any positive integer.

(iv) (x + a) is a factor of $(x^n + a^n)$ where "n" is an odd positive integer.

(v) (x + a) is a factor of $(x^n - a^n)$ where "n" is positive even integer.

(vi) $(x^n + a^n)$ is not divisible by (x + a) when "n" is even integer.

(vii) $(x^n + a^n)$ is not divisible by (x – a) for any integer "n"

(viii) If (x – 1) is a factor of polynomial of degree 'n' then the condition is sum of the coefficients is zero.

(ix) If (x + 1) is a factor of polynomial of degree 'n' then the condition is the sum of the coefficients of even terms is equal to the sum of the coefficients of odd terms.

Ex.1: Use factor theorem to verify that (x + a) is a factor of $(x^n + a^n)$ for any odd positive integer n.

Sol: $p(x) = x^n + a^n$

$p(-a) = (-a)^n + a^n$

for any odd positive integer n, $(-a)^n = -a^n$

therefore, $p(-a) = -a^n + a^n$

$p(-a) = 0.$

yes, $(x + a)$ is a factor of $x^n + a^n$ for any odd positive integer n.

Ex.2: Show that $(x-3)$ is a factor of the polynomial $f(x) = x^3 + x^2 - 17x + 15$.

Sol: By the factor theorem, $(x-3)$ will be a factor of $f(x)$ if $f(3) = 0$

Now, $f(x) = x^3 + x^2 - 17x + 15$

$f(3) = (3^3 + 3^2 - 17 \times 3 + 15) = (27 + 9 - 51 + 15) = 0$

Hence $(x-3)$ is a factor of the given polynomial $f(x)$.

Ex.3: Find the value of a for which $(x + a)$ is a factor of the polynomial

$f(x) = x^3 + ax^2 - 2x + a + 6.$

Sol: $(x + a)$ is a factor of $f(x) = x^3 + ax^2 - 2x + a + 6$

$\Rightarrow \quad f(-a) = 0$

$\Rightarrow \quad (-a)^3 + a(-a)^2 - 2(-a) + a + 6 = 0$

$\Rightarrow \quad 3a = -6 \quad \Rightarrow \quad a = -2$

Hence, the required value of a is -2.

Ex.4: Using remainder theorem show that $(a - b)$, $(b - c)$ and $(c - a)$ are the factors of the $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Sol: $p(a) = a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

if $(a - b)$ is a factor of $p(a)$ then remainder

$= p(b) = b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3$

Remainder $= 0$

Since remainder $= 0$, therefore $(a - b)$ is a factor of

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Similarly,

if polynomial is p(b) then remainder = p(c) = 0

if polynomial is p(c) then remainder = p(a) = 0

therefore (b – c) and (c – a) are also factors of

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.