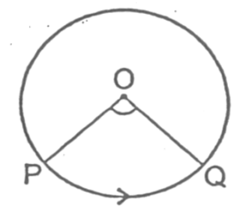
CIRCLES

ANGLE SUBTENDED BY AN ARC OF A CIRCLE

Angle Subtended By An Arc Of A Circle:

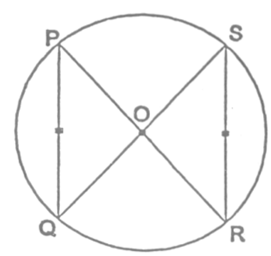
In figure, the angle subtended by the minor arc PQ at O is ∠POQ and the angle subtended by the major arc PQ at O is reflex angle ∠POQ.



Theorem 1:

Equal chords of a circle subtend equal angles at the centre.

Given : A circle with centre O in which chord PQ = chord RS.



To Prove : ∠POQ = ∠ROS.

Proof :

In ΔPOQ and ΔROS,

OP = OR [Radii of the same circle]

OQ = OS [Radii of the same circle]

PW = RS [Given]

⇒ ΔPOQ = ΔROS [By SSS]

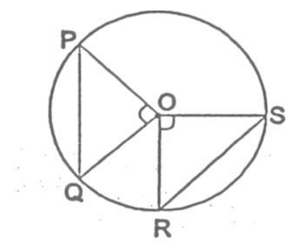
⇒ ∠POQ = ∠ROS [By cpctc] **Hence Proved.**

Theorem-2 :

If the angles subtended by the chords at the centre (of a circle) are equal then the chords are equal.

Given : A circle with centre O . Chords PQ and RS subtend equal angles at the enter of the circle. i.e. $\angle POQ = \angle ROS$

To Prove : Chord PQ = chord RS.



Proof :

In ΔPOQ and ΔROS,

$\angle POQ = \angle ROS$ [Given]

$OP = OR$ [Radii of the same circle]

$OQ = OS$ [Radii of the same circle]

$\Rightarrow$ $\Delta POQ \cong \Delta ROS$ [By SSS]

$\Rightarrow$ chord PQ = chord RS [By cpctc] **Hence Proved.**

Corollary-1 :

Two arc of a circle are congruent, if the angles subtended by them at the centre are equal.

Corollary 2 :

If two arcs of a circle are equal, they subtend equal angles at the centre.

Corollary 3 :

If two arc of a circle are congruent (equal), their corresponding chords are equal.

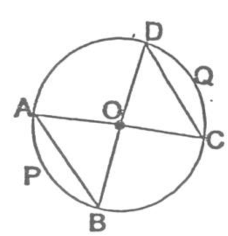
Corollary 4:

If two chords of a circle are equal, their corresponding arc are also equal.

$\angle AOB = \angle COD$

$\therefore$ Chord AB = Chord CD

$\therefore$ Arc APB = Arc COD.



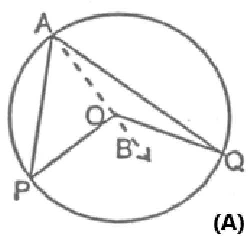
Theorem-3 :

The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

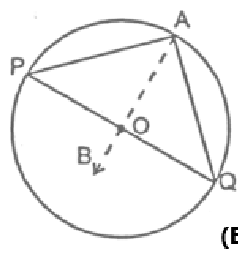
Given : An arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle.

To Prove : $\angle POQ = 2\angle PAQ$.

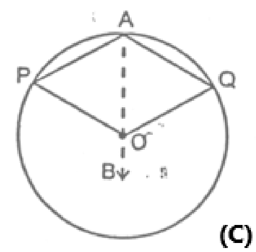
Construction : Join AO and extend it to a point B.



(A)



(B)



(C)

Proof :

There arises three cases :

(A) are PQ is minor

(B) arc PQ s a semi - circle

(C) arc PQ is major.

In all the cases,

$\angle BOQ = \angle OAQ + \angle AQO$ ....(i)

[∵An exterior angle of triangle is equal to the sum of the two interior opposite angles]

In OAQ,

OA = OQ [Radii of a circle]

$\therefore$ $\angle OAQ = \angle OQA$ ...(ii) [Angles opposite equal of a triangle are equal]

(i) and (ii), give,

$\angle BOQ = 2\angle OAQ$ ....(iii)

Similarly,

$\angle BOP = 2\angle OAP$ ....(iv)

Adding (iii) and (iv), we get

$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$

$\Rightarrow \quad \angle POQ = 2\angle PA.$ ....(v)

NOTE : For the case (C), where PQ is the major arc, (v) is replaced by reflex angles.

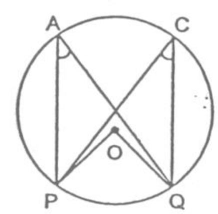
Thus, $\angle POQ = 2\angle PAQ.$

Theorem- 4 :

Angles in the same segment of a circle are equal.

Proof :

Let P and Q be any two points on a circle to form a chord PQ, A and C any other points on the remaining part of the circle and O be the centre of the circle. Then,



$\angle POQ = 2\angle PAQ$ ...(i)

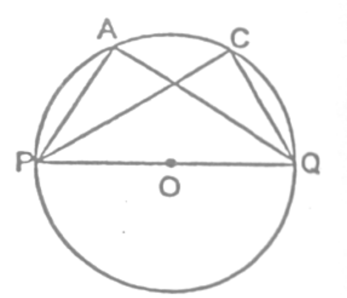
And $\angle POQ = 2\angle PCQ$ ...(ii)

From above equations, we get

$2\angle PAQ = 2\angle PCQ$

$\Rightarrow \quad \angle PAQ = \angle PCQ$ **Hence Proved**

Theorem-5 :



Angle in the semicircle is a right angle.

Proof :

$\angle PAQ$ is an angle in the segment, which is a semicircle.

$\therefore \quad \angle PAQ = \frac{1}{2}\angle PAO = \frac{1}{2} \times 180^0 = 90^0$

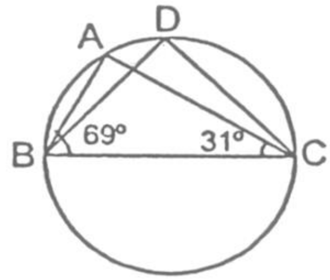
[$\therefore$ $\angle PQR$ is straight line angle or $\angle PQR = 180^0$]

If we take any other point C on the semicircle, then again we get

$\angle PCQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180^0 = 90^0$ **Hence Proved.**

Ex.1 In figure, $\angle ABC = 69^0$, $\angle ACB = 31^0$, find $\angle BDC$.

Sol. In ΔABC.



$\angle BAC + \angle ABC + \angle ACB = 180^0$

[Sum of all the angles of a triangle is 180^0]

$\Rightarrow \angle BAC + 69^0 + 31^0 = 180^0$

$\Rightarrow \angle BAC + 100^0 = 180^0$

$\Rightarrow \angle BAC = 180^0 - 100^0 = 80^0$

Now, $\angle BDC = \angle BAC = 80^0$.

Ans. [Angles in the same segment of a circle are equal]

Theorem-6:

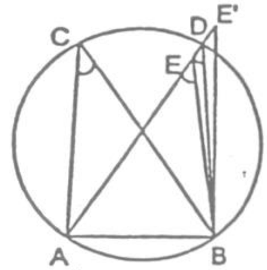
If a line segment joining two points subtend equal angles at two other points lying on the same side of the lien containing the line segment the four points lie on a circle (i.e., they are concyclic).

Given : AB is a line segment, which subtends equal angles at two points C and D.

i.e., $\angle ACB = \angle ADB$.

To Prove : The points A, B, C and D lie on a circle.

Proof :



Let us draw a circle through the points A, C and B.

Suppose it does not pass through the point D.

Then it will intersect AD (or extended AD) at a point, say E (or E').

If points A,C,E and B lie on a circle,

$\angle ACD = \angle AEB$ [$\therefore$ Angles in the same segment of circle are equal]

But it is given that $\angle ACB = \angle ADB$

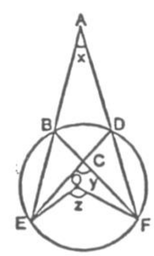
Therefore, $\angle AEB = \angle ADB$

This is possible only when E coincides with D. [As otherwise $\angle AEB > \angle ADB$]

Similarly, E' should also coincide with D. So A, B, C and D are concyclic **Hence Proved.**

Ex.2 In figure, O is the centre of the circle. Prove that $\angle x + \angle y = \angle z$.

Sol. $\angle EBF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$



[$\because$Angle subtended by an arc of a circle at the centre in twice the angle subtended by it at any point of the remaining part of the circle]

$\therefore$ $\angle ABF = 180^0 - \frac{1}{2}\angle z$ ...(i) [Linear Pair Axiom]

$\angle EDF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$

[$\because$Angle subtend by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$\therefore$ $\angle ADE = 180^0 - \frac{1}{2}\angle z$ ....(ii) [Linear Pair Axiom]

$\angle BCD = \angle ECF = \angle y$ [Vert. Opp. Angle]

$\angle BAD = \angle x$

In quadrilateral ABCD

$\angle ABC + \angle BCD + \angle CDA + \angle BAD = 2 \times 180^0$

[Angle Sum Property of a quadrilateral]

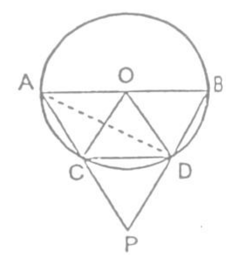
$\Rightarrow\ 180^0 - \frac{1}{2}\angle z + \angle y + 180^0 - \frac{1}{2}\angle z + \angle x = 2 \times 180^0$

$\Rightarrow\ \angle x + \angle y = \angle z$ **Hence Proved.**

Ex.3 AB is a diameter of the circle with centre O and chord CD is equal to radius OC, AC and BD produced meet at P. Prove that $\angle CPD = 60^0$.

Sol. **Given :** AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P.

To Prove : $\angle CPD = 60^0$



Construction : Join AD.

Proof :

In ΔOCD,

$OC = OD$ ...(i) [Radii of the same circle]

$OC = CD$ ....(ii) [Given]

From (i) and (ii),

$OC = OD = CD$

$\therefore\ \Delta OCD$ is equilateral

$\therefore\ \angle COD = 60^0$

$\therefore\ \angle CAD = \frac{1}{2}\angle COD = \frac{1}{2}\angle(60^0) = 30^0$

[$\because$Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the reaming part of the circle]

$\Rightarrow\ \angle PAD = 30^0$ .....(iii)

And, $\angle ADB = 90^0$ .....(iv) [Angle in a semi-circle]

$\Rightarrow \angle ADB + \angle ADP = 180^0$ [Linear Pair Axiom]

$\Rightarrow 90^0 + \angle ADP = 180^0$ [From (iv)]

$\Rightarrow \angle ADP = 90^0$ ....(v)

In ΔDP,

$\angle ADP + \angle PAD + \angle ADP = 180^0$

[$\because$The sum of the three angles of a triangles is 180^0]

$\Rightarrow \angle APD + 30^0 + 90^0 = 180^0$ [From (iii) and (v)]

$\Rightarrow \angle APD + 120^0 = 180^0$

$\Rightarrow \angle APD = 180^0 - 120^0 = 60^0$

$\Rightarrow \angle CPD = 60^0$. **Hence Proved.**