# NUMBER SYSTEMS

## SURDS AND RADICALS

## SURDS :

Any irrational number of the form $\sqrt[n]{a}$ is given a special name surd. Where 'a' is called radicand, it should always be a rational number. Also the symbol $\sqrt[n]{\ }$ is called the radical sign and the index n is called order of the surd.

$\sqrt[n]{a}$ is read as 'n$^{th}$ **root a'** and can also be written as $a^{\frac{1}{n}}$.

### (a) Some Identical Surds :

(i) $\sqrt[3]{4}$ is a surd as radicand is a rational number.

Similar examples $\sqrt[3]{5}, \sqrt[4]{12}, \sqrt[5]{7}, \sqrt{12}, \ldots\ldots.$

(i) $2\sqrt{3}$ is a surd (as surd + rational number will give a surd)

Similar examples $\sqrt{3}+1, \sqrt[3]{3}+1, \ldots$

(iii) $\sqrt{7-4\sqrt{3}}$ is a surd as 7 - $4\sqrt{3}$ is a perfect square of $\left(2-\sqrt{3}\right)$

Similar examples $\sqrt{7+4\sqrt{3}}, \sqrt{9-4\sqrt{5}}, \sqrt{9+4\sqrt{5}}, \ldots\ldots.$

(i) $\sqrt[3]{\sqrt{3}}$ is a surd as $\sqrt[3]{\sqrt{3}} = \left(3^{\frac{1}{2}}\right)^{\frac{1}{3}} = 3^{\frac{1}{6}} = \sqrt[6]{3}$

Similar examples $\sqrt[3]{\sqrt[3]{5}}, \sqrt[4]{\sqrt[5]{6}}, \ldots\ldots..$

## (b) Some Expression are not Surds :

(i) $\sqrt[3]{8}$ because $\sqrt[3]{8}=\sqrt[3]{2^3}=2,$ which is a rational number.

(ii) $\sqrt{2+\sqrt{3}}$ because $2+\sqrt{3}$ is not a perfect square.

(iii) $\sqrt[3]{1+\sqrt{3}}$ because radicand is an irrational number.

## LAW OF SURDS:

(i) $\left(\sqrt[n]{a}\right)^n=\sqrt[n]{a^n}=a$

e.g. (A) $\sqrt[3]{8}=\sqrt[3]{2^3}=2$ (B) $\sqrt[4]{81}=\sqrt[4]{3^4}=3$

(ii) $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$ [Here order should be same]

e.g. (A) $\sqrt[3]{2}\times\sqrt[3]{6}=\sqrt[3]{2\times 6}=\sqrt[3]{12}$

but, $\sqrt[3]{3}\times\sqrt[4]{6}\neq\sqrt{3\times 6}$ [Because order is not same]

1st make their order same and then you can multiply.

(iii) $\sqrt[n]{a}+\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}$

(iv) $\sqrt[n]{\sqrt[m]{a}}=\sqrt[nm]{a}=\sqrt[m]{\sqrt[n]{a}}$ e.g. = $\sqrt{\sqrt{\sqrt{2}}}=\sqrt[8]{8}$

(v) $\sqrt[n]{a}=\sqrt[n\times p]{a^p}$ [Important for changing order of surds]

or, $\sqrt[n]{a^m}=\sqrt[n\times p]{a^{m\times p}}$

e.g. $\sqrt[3]{6^2}$ make its order 6, then $\sqrt[3]{6^2}=\sqrt[3\times 2]{6^{2\times 2}}=\sqrt[6]{6^4}$.

e.g. $\sqrt[3]{6}$ make its order 15, then $\sqrt[3]{6}=\sqrt[3\times 5]{6^{1\times 5}}=\sqrt[15]{6^5}$.

## OPERATION OF SURDS:

### (a) Addition and Subtraction of Surds :

Addition and subtraction of surds are possible only when order and radicand are same i.e. only for surds.

**Ex.1** Simplify

(i) $\sqrt{6}-\sqrt{216}+\sqrt{96}=15\sqrt{6}-\sqrt{6^2\times 6}+\sqrt{16\times 6}$

(ii) $5\sqrt[3]{250}+7\sqrt[3]{16}-14\sqrt[3]{54}$

**Sol.**(i) $\sqrt{6}-\sqrt{216}+\sqrt{96}=15\sqrt{6}-\sqrt{6^2\times 6}+\sqrt{16\times 6}$ [Bring surd in simples form]

$$= 15\sqrt{6}-6\sqrt{6}+4\sqrt{6}$$

$$= (15 - 6 + 4)\sqrt{6}$$

$$= 13\sqrt{6}$$

(ii) $5\sqrt[3]{250}+7\sqrt[3]{16}-14\sqrt[3]{54} = 5\sqrt[3]{125\times 2}+7\sqrt[3]{8\times 2}-14\sqrt[3]{27\times 2}$

$$=5\times 5\sqrt[3]{2}+7\times 2\sqrt[3]{2}-14\times 3\times\sqrt[3]{2}$$

$$=(25+14-42)\sqrt[3]{2}$$

$$=-3\sqrt[3]{2}$$

(iii) $4\sqrt{3}+3\sqrt{48}-\frac{5}{2}\sqrt{\frac{1}{3}}=4\sqrt{3}+3\sqrt{16\times 3}-\frac{5}{2}\sqrt{\frac{1\times 3}{3\times 3}}$

$$=4\sqrt{3}+3\times 4\sqrt{3}-\frac{5}{2}\times\frac{1}{3}\sqrt{3}$$

$$=4\sqrt{3}+12\sqrt{3}-\frac{5}{6}\sqrt{3}$$

$$=\left(4+12-\frac{5}{6}\right)\sqrt{3} \quad =\frac{91}{6}\sqrt{3}$$

## (b) Multiplication and Division of Surds :

**Ex.2** (i) $\sqrt[3]{4} \times \sqrt[3]{22}$ (ii) $\sqrt[3]{2} \times \sqrt[4]{3}$

**Sol.** (i) $\sqrt[3]{4} \times \sqrt[3]{22} = \sqrt[3]{4 \times 22} = \sqrt[3]{2^3 \times 11} = 2\sqrt[3]{11}$

(ii) $\sqrt[3]{2} \times \sqrt[4]{3} = \sqrt[12]{2^4} \times \sqrt[12]{3^3} = \sqrt[12]{2^4 \times 3^3} = \sqrt[12]{16 \times 27} = \sqrt[12]{432}$

**Ex.3** Simplify $\sqrt{8a^5b} \times \sqrt[3]{4a^2b^2}$

**Sol:** $\sqrt[6]{8^3a^{15}b^3} \times \sqrt[6]{4^2a^4b^4} = \sqrt[6]{2^{13}a^{19}b^7} = \sqrt[6]{2ab}.$

**Ex.4** $\sqrt{\mathbf{24}} \div \sqrt[3]{\mathbf{200}}$

**Sol.** $\sqrt{24} \div \sqrt[3]{200} = \frac{\sqrt{24}}{\sqrt[3]{200}} = \frac{\sqrt[6]{(24)^3}}{\sqrt[6]{(200)^2}} = \sqrt[6]{\frac{216}{625}}$

## (c) Comparison of Surds :

It is clear that if $x > y > 0$ and $n > 1$ is a positive integer then $\sqrt[n]{x} > \sqrt[n]{y}$.

**Ex.6** Arrange $\sqrt{2}, \sqrt[3]{3}$ and $\sqrt[4]{5}$ is ascending order.

**Sol.** L.C.M. of 2, 3, 4 is 12.

$\therefore \sqrt{2} = \sqrt[2\times6]{2^6} = \sqrt[12]{64}$

$\sqrt[3]{3} = \sqrt[3\times4]{3^4} = \sqrt[12]{81}$

$\sqrt[4]{5} = \sqrt[4\times3]{5^3} = \sqrt[12]{125}$

As, $64 < 81 < 125$.

$\therefore \sqrt[12]{64} < \sqrt[12]{81} < \sqrt[12]{125}$

$\Rightarrow \sqrt{2} < \sqrt[3]{3} < \sqrt[4]{5}$

**Ex.7** Which is greater $\sqrt{7}-\sqrt{3}$ or $\sqrt{5}-1$?

**Sol.** $\sqrt{7}-\sqrt{3}=\frac{(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})}{(\sqrt{7}+\sqrt{3})}=\frac{7-3}{\sqrt{7}+\sqrt{3}}=\frac{4}{\sqrt{7}+\sqrt{3}}$

And, $\sqrt{5}-1=\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{(\sqrt{5}+1)}=\frac{5-1}{\sqrt{5}+1}=\frac{4}{\sqrt{5}+1}$

Now, we know that $\sqrt{7}>\sqrt{5}$ and $\sqrt{3}>1$, add

So, $\sqrt{7}+\sqrt{3}>\sqrt{5}+1$

$\Rightarrow \frac{1}{\sqrt{7}+\sqrt{3}}<\frac{1}{\sqrt{5}+1}$

$\Rightarrow \frac{4}{\sqrt{7}+\sqrt{3}}<\frac{4}{\sqrt{5}+1}$

$\Rightarrow \sqrt{7}-\sqrt{3}<\sqrt{5}-1$

So, $\sqrt{5}-1>\sqrt{7}-\sqrt{3}$

## RATIONALIZATION OF SURDS:

Rationalizing factor product of two surds is a rational number then each of them is called the rationalizing factor (R.F.) of the other. The process of converting a surd to a rational number by using an appropriate multiplier is known as rationalization.

### Some examples:

(i) R.F. of $\sqrt{a}$ is $\sqrt{a}$ $\left(\because \sqrt{a}\times\sqrt{a}=a\right)$.

(ii) R.F. of $\sqrt[3]{a}$ is $\sqrt[3]{a^2}$ $\left(\because \sqrt[3]{a}\times\sqrt[3]{a^2}=\sqrt[3]{a^3}=a\right)$.

(iii) R.F. of $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$ & vice versa $\left[\because (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b\right]$.

(iv) R.F. of a $+\sqrt{b}$ is $a-\sqrt{b}$ & vice versa $\left[\because (a+\sqrt{b})(a-\sqrt{b})=a^2-b\right]$

(v) R.F. of $\sqrt[3]{a}+\sqrt[3]{b}$ is $\left(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}\right)\left[\because (\sqrt[3]{a}+\sqrt[3]{b})\left(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}\right)\right]$

$\left[\therefore \left(\sqrt[3]{a}\right)^3+\left(\sqrt[3]{b}\right)^3=a+b\right]$ which is rational.

(vi) R.F. of $\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$ is $\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)$ and $\left(a+b-c+2\sqrt{ab}\right)$

**Ex.8** Find the R.G. (rationalizing factor) of the following :

(i) $\sqrt{10}$ (ii) $\sqrt{12}$ (iii) $\sqrt{162}$

(iv) $\sqrt[3]{4}$ (v) $\sqrt[3]{16}$ (vi) $\sqrt[4]{162}$

(vii) $2+\sqrt{3}$ (viii) $7-4\sqrt{3}$ (ix) $3\sqrt{3}+2\sqrt{2}$

(x) $\sqrt[3]{3}+\sqrt[3]{2}$ (xi) $1+\sqrt{2}+\sqrt{3}$

**Sol.** (i) $\sqrt{10}$

$[\therefore \sqrt{10}\times\sqrt{10}=\sqrt{10\times10}=10]$ as 10 is rational number.

$\therefore$ R.F. of $\sqrt{10}$ is $\sqrt{10}$

(ii) $\sqrt{12}$

First write it's simplest from i.e. $2\sqrt{3}$.

Now find R.F. (i.e. R.F. of $\sqrt{3}$ is $\sqrt{3}$)

$\therefore$ R.F. of $\sqrt{12}$ is $\sqrt{3}$

(iii) $\sqrt{162}$

Simplest from of $\sqrt{162}$ is $9\sqrt{2}$.

R.F. of $\sqrt{2}$ is $\sqrt{2}$.

$\therefore$ R.F. of $\sqrt{162}$ is $\sqrt{2}$

(iv) $\sqrt[3]{4}$

$\sqrt[3]{4}\times\sqrt[3]{4^2}=\sqrt[3]{4^3}=4$

$\therefore$ R.F. of $\sqrt[3]{4}$ is $\sqrt[3]{4^2}$

(v) $\sqrt[3]{16}$

Simplest from of $\sqrt[3]{16}$ is $2\sqrt[3]{2}$

Now R.F. of $\sqrt[3]{2}$ is $\sqrt[3]{2^2}$

$\therefore$ R.F. of $\sqrt[3]{16}$ is $\sqrt[3]{2^2}$

(vi) $\sqrt[4]{162}$

Simplest form of $\sqrt[4]{162}$ is $3\sqrt[4]{2}$

Now R.F. of $\sqrt[4]{2}$ is $\sqrt[4]{2^3}$

R.F. of $\left(\sqrt[4]{162}\right)$ is $\sqrt[4]{2^3}$

(vii) $2+\sqrt{3}$

As $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=(2)^2-\left(\sqrt{3}\right)^2=4-3=1$, , which is rational.

$\therefore$ R.F. of $\left(2+\sqrt{3}\right)$ is $\left(2-\sqrt{3}\right)$

(viii) $7-4\sqrt{3}$

As $\left(7-4\sqrt{3}\right)\left(7+4\sqrt{3}\right)=(7)^2-\left(4-\sqrt{3}\right)^2$ = 49 - 48 = 1, which is rational

$\therefore$ R.F. of $\left(7-4\sqrt{3}\right)$ is $\left(7+4\sqrt{3}\right)$

(ix) $3\sqrt{3}+2\sqrt{2}$

As $\left(3\sqrt{3}+2\sqrt{2}\right)\left(3\sqrt{3}-2\sqrt{2}\right)=\left(3\sqrt{3}\right)^2-\left(2\sqrt{2}\right)^2=27-8=19$, which is rational.

$\therefore$ R.F. of $\left(3\sqrt{3}+2\sqrt{2}\right)$ is $\left(3\sqrt{3}-2\sqrt{2}\right)$

(x) $\sqrt[3]{3}+\sqrt[3]{2}$

As $\left(\sqrt[3]{3}+\sqrt[3]{2}\right)\left(\sqrt[3]{3^2}-\sqrt[3]{3}\times\sqrt[3]{2}+\sqrt[3]{2^2}\right)=\left(\sqrt[3]{3^3}+\sqrt[3]{2^3}\right)$ = 3 + 2 = 5, which is rational.

$\therefore$ R.F. of $(\sqrt[3]{3}+\sqrt[3]{2})$ is $\left(\sqrt[3]{3^2}-\sqrt[3]{3}\times\sqrt[3]{2}+\sqrt[3]{2^2}\right)$

(xi) $1+\sqrt{2}+\sqrt{3}$

$$(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})=(1+\sqrt{2})^2-(\sqrt{3})^2$$

$$=1)^2+(\sqrt{2})^2+2(1)(\sqrt{2})-3$$

$$=1+2+2\sqrt{2}-3$$

$$=3+2\sqrt{2}-3$$

$$=2\sqrt{2}$$

$$2\sqrt{2}\times\sqrt{2}=2\times2=4$$

$\therefore$ R.F. of $1+\sqrt{2}+\sqrt{3}$ is $(1+\sqrt{2}-\sqrt{3})$ and $\sqrt{2}$.

**NOTE :** R.F. of $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$ type surds are also called conjugate surds & vice versa.

**Ex.9** Express the following surd with a rational denominator.

**Sol.** $$\frac{8}{\sqrt{15}+1-\sqrt{5}-\sqrt{3}}=\left[\frac{8}{(\sqrt{15}+1)-(\sqrt{15}+\sqrt{3})}\right]\times\left[\frac{(\sqrt{15}+1)+(\sqrt{5}+\sqrt{3})}{(\sqrt{15}+1)+(\sqrt{5}+\sqrt{3})}\right]$$

$$=\frac{8(\sqrt{15}+1+\sqrt{5}+\sqrt{3})}{(\sqrt{15}+1)^2-(\sqrt{5}+\sqrt{3})^2}$$

$$=\frac{8(\sqrt{15}+1+\sqrt{5}+\sqrt{3})}{15+1+2\sqrt{15}-(5+3+2\sqrt{15})}$$

$$=\frac{8(\sqrt{15}+1+\sqrt{5}+\sqrt{3})}{8}$$

$$=(\sqrt{15}+1+\sqrt{5}+\sqrt{3})$$

**Ex.10** Rationalize the denominator of $\dfrac{a^2}{\sqrt{a^2+b^2}+b}$

**Sol.** $\dfrac{a^2}{\sqrt{a^2+b^2}+b}=\dfrac{a^2}{\sqrt{a^2+b^2}+b}\times\dfrac{\sqrt{a^2+b^2}-b}{a^2+b^2-b}$

$$=\frac{a^2\left(\sqrt{a^2+b^2}-b\right)}{\left(\sqrt{a^2+b^2}\right)^2-(b)^2}$$

$$=\frac{a^2\left(\sqrt{a^2+b^2}-b\right)}{a^2+b^2-b^2}=\left(\sqrt{a^2+b^2}-b\right)$$

## EXPONENTS OF REAL NUMBER

### (a) Positive Integral Power :

For any real number a and a positive integer 'n' we define $a^n$ as :

$a^n = a \times a \times a \times$ ............... x a (n times)

$a^n$ is called then $n^{th}$ power of a. The real number 'a' is called the base and 'n' is called the exponent of the $n^{th}$ power of a.

e.g. $2^3 = 2 \times 2 \times 2 = 8$

**NOTE :** For any non-zero real number 'a' we define $a^0 = 1$.

e.g. thus, $3^0 = 1$, $5^0$, $\left(\dfrac{3}{4}\right)^0 = 1$ and so on.

### (b) Negative Integral Power :

For any non-zero real number 'a' and a positive integer 'n' we define $a^{-n}=\dfrac{1}{a^n}$

Thus we have defined $a^n$ find all integral values of n, positive, zero or negative. $a^n$ is called the $n^{th}$ power of a.

## RATIONAL EXPNENTS OR A REAL NUMBER

### (a) Principal of $n^{th}$ Root of a Positive Real Numbers :

If 'a' is a positive real number and 'n' is a positive integer, then the principal $n^{th}$ root of a is the unique positive real number x such that $x^n = a$.

The principal $n^{th}$ root of a positive real number a is denoted by $a^{1/n}$ or $\sqrt[n]{a}$.

### (b) Principal of $n^{th}$ Root of a Negative Real Numbers :

If 'a' is a negative real number and 'n' is an odd positive integer, then the principle $n^{th}$ root of a is define as $-|a|^{1/n}$ i.e. the principal $n^{th}$ root of -a is negative of the principal $n^{th}$ root of $|a|$.

### REMAEK:

It 'a' is negative real number and 'n' is an even positive integer, then the principle $n^{th}$ root of a is not defined, because an even power of real number is always positive. Therefore $(-9)^{1/2}$ is a meaningless quantity, if we confine ourselves to the set of real number, only.

### (c) Rational Power (Exponents) :

For any positive real number 'a' and a rational number $\frac{p}{q} \neq$ where $q \neq 0$, we define

$a^{p/q} = (a^p)^{1/q}$ i.e. $a^{p/q}$ is the principle $q^{th}$ root of $a^p$.

### LAWS OF RATIONAL EXPONETNS

The following laws hold the rational exponents

(i) $a^m \times a^n = a^{m+1}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $a^{-n} = \frac{1}{a^n}$

(v) $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$ i.e. $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

(vi) $(ab)^m = a^m b^m$

(vii) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ (viii) $a^{bn} = a^{b+b+b\ldots.n\text{ tmes}}$

Where a,b are positive real number and m,n are relational numbers.

## ILLUSTRATIONS :

**Ex.11** Evaluate each of the following:

(i) $5^2 \times 5^4$ (ii) $5^8 \div 5^3$ (iii) $(3^2)^2$

(iv) $\left(\frac{11}{12}\right)^3$ (v) $\left(\frac{3}{4}\right)^{-3}$

**Sol.** Using the laws of indices, we have

(i) $5^2.5^4 = 5^{2+4} = 5^6 = 15625$ $\because a^m \times a^n = a^{m+n}$

(ii) $5^8 \div 5^3 = \frac{5^8}{5^3} = 5^{8-3} = 5^5 = 3125$ $\because a^m + a^n = a^{m-n}$

(iii) $(3^2)^3 = 3^{2\times3} = 3^6 = 729$ $\because (a^m)^n = a^{m\times n}$

(iv) $\left(\frac{11}{12}\right)^3 = \frac{11^3}{12^3} = \frac{1331}{1728}$ $\because \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

(v) $\left(\frac{3}{4}\right)^{-3} = \frac{1}{\left(\frac{3}{4}\right)^3} = \frac{1}{\frac{3^3}{4^3}} = \frac{1}{\frac{27}{64}} = \frac{64}{27}$ $\because a^{-n} = \frac{1}{a^n}$

**Ex.12** Evaluate each of the following :

(i) $\left(\frac{2}{11}\right)^4 \times \left(\frac{11}{3}\right)^2 \times \left(\frac{3}{2}\right)^3$ (ii) $\left(\frac{1}{2}\right)^5 \times \left(\frac{-2}{3}\right)^4 \times \left(\frac{3}{5}\right)^{-1}$

(iii) $2^{55} \times 2^{60} - 2^{97} \times 2^{18}$ (iv) $\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^2$

**Sol.** We have.

(i) $\left(\frac{2}{11}\right)^4 \times \left(\frac{11}{3}\right)^2 \times \left(\frac{3}{2}\right)^3 = \frac{2^4}{11^4} \times \frac{11^2}{3^2} \times \frac{3^3}{2^3}$

$= \frac{2 \times 3}{11^2} \qquad = \frac{6}{121}$

(ii) We have,

$\left(\frac{1}{2}\right)^5 \times \left(\frac{-2}{3}\right)^4 \times \left(\frac{3}{5}\right)^{-1} = \left(\frac{1}{2}\right)^5 \times \left(\frac{-2}{3}\right)^4 \times \left(\frac{1}{\frac{3}{5}}\right)$

$= \frac{1^5}{2^5} \times \frac{(-2)^2}{3^4} \times \frac{5}{3}$

$= \frac{1 \times 16 \times 5}{32 \times 81 \times 3}$

$= \frac{5}{2 \times 81 \times 3} \qquad = \frac{5}{486}$

(iii) We have,

$2^{55} \times 2^{60} - 2^{97} \times 2^{18} = 2^{55+60} - 2^{97+18}$

$= 2^{15} - 2^{115} \qquad = 0$

(iv) We have,

$\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{5}\right)^{-3} \times \left(\frac{3}{5}\right)^{-2} = \frac{2^3}{3^3} \times \frac{1}{(2/5)} \times \frac{3^2}{5^2}$

$= \frac{2^3}{3^3} \times \frac{1}{2^3/5^3} \times \frac{3^2}{5^2}$

$= \frac{2^3 \times 5^3 \times 3^2}{3^3 \times 2^3 \times 5^2} \qquad = \frac{5}{3}$