INTEGRALS

5.1 BASIC CONCEPTS

If f(x) is derivative of function g(x), then g(x) is known as antiderivative or integral of f(x).

i.e., $\frac{d}{dx}\{g(x)\} = f(x) \Leftrightarrow \int f(x)dx = g(x)$.

(a) Derivative of a function is unique but a function can have infinite antiderivatives or integrals.

(b) $\int f(x)dx = g(x) + c$, where c is constant of integration, is knwon as indefinite integral.

(c) $\int_a^b f(x)dx = \{g(x) + c\}_a^b = g(b) - g(a)$. **(d)** $\int dx = x + c$. **(e)** $\int c.f(x)dx = c.\int f(x)dx$.

(f) $\int \{(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$.

(g) $\int f(x)dx \Leftrightarrow \int f\{g(t)\}.\ g'(t)dt$, if we substitute x = g(t) such that dx = g′(t)dt.

5.2 STANDARD RESULT

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$, n is a rational number. If n = –1, then $\int \frac{1}{x} dx = \log|x| + c$.

2. $\int (ax + b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$, n is a rational number.

If n = – 1. Then $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax + b| + c$.

3. In case of rational function, if degree of numerator is equal or greater than degree of denominator, then we first divide numerator by denominator and write it as

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}},$$ and then integrate.

4. $\int \sin ax\, dx = \frac{-\cos ax}{a} + c$. If a = 1, $\int \sin x\, dx = -\cos x + c$.

5. $\int \cos ax\, dx = \frac{\sin ax}{a} + c$. If a = 1, $\int \cos x\, dx = \sin x + c$.

6. $\int \tan ax\, dx = -\frac{1}{a}\log|\cos as| + c$. or $\frac{1}{a} \log|\sec ax| + c$.

If a = 1, then $\int \tan x\, dx = -\log|\cos x| + c$ or $\log|\sec x| + c$.

7. $\int \cot ax\, dx = \frac{1}{a}\log|\sin ax| + c$. If a = 1, then $\int \cot x\, dx = \log|\sin x| + c$.

8. $\int \sec ax\, dx = \frac{1}{a}\log|\sec ax + \tan ax| + c$. If a = 1, then $\int \sec x\, dx = \log|\sec x + \tan x| + c$.

9. $\int \text{cosec}\, ax\, dx = \frac{1}{a}\log|\text{cosec}\, ax - \cot ax| + c$. If a = 1, then $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + c$.

10. $\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + c$. If a = 1, then $\int \sec x \tan x\, dx = \sec x + c$.

11. $\int \text{cosec}\, ax \cot ax\, dx = -\frac{1}{a}\text{cosec}\, ax + c$. If a = 1, then $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$.

12. $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + c$. If a = 1, then $\int \sec^2 x\, dx = \tan x + c$.

13. $\int \text{cosec}^2 ax\, dx = -\frac{1}{a}\cot ax + c$. If a = 1, then $\int \text{cosec}^2 x\, dx = -\cot x + c$.

14. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$. If a = 1, then $\int e^x dx = e^x + c$.

15. $\int a^{mx} dx = \frac{a^{mx}}{m\log_e a} + c$. If m = 1, then $\int a^x dx = \frac{a^x}{\log_e a} + c$.

16. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c$. If a = 1, then $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c$.

17. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$. If a = 1, then $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$.

18. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$. If a = 1, then $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}x + c$.

19. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log\left|x + \sqrt{a^2+x^2}\right| + c$.

20. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$.

21. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$.

22. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$.

23. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$.

24. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$.

25. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$.

5.3 METHODS OF INTEGRATION

(i) Integration by Substitution (or change of independent variable) :

If the independent variable x in $\int f(x)\, dx$ be changed to t, then we substitute $x = \phi(t)$

i.e., $dx = \phi'(t)\, dt$ $\quad \int f(x)\, dx = \int f(\phi(t))\phi'(t)\, dt$

which is either a standard form or is easier to integrate.

(ii) Integration by parts :

If u and v are the differentiable functions of x then $\int u.\, v\, dx = u\int v dx - \int\left[\left(\frac{d}{dx}u\right)\left(\int v dx\right)\right] dx$.

How to choose Ist and IInd functions:

(a) If the two functions are of different types take that function as Ist which comes first in the word **ILATE** where I stands for inverse circular function, L stands for logarithrmic function A stands for Algebraic function, T stands for trigonometric function and E stands for exponential function.

(b) If both functions are algebraic take that function as Ist whose differential coefficient is simple.

(c) It both functions are trigonometrical take that function as IInd whose integral is simpler.

(d) Successive integration by parts : Use the following formula

$$\int uv dx = uv_1 - u'v_2 + u''v_3 - u''' v_4 + \dots\dots\dots + (-1)^{n-1} u^{n-1} v_n + (-1)^n \int u^n v_n dx.$$

where u^n stands for nth differential coefficient of u w. r. t. x and v_n stands for n th integral of v w. r. t. x.

cancellation of Integrals : i.e. $\int e^x\{(f(x) + f'(x)\}\, dx = e^x f(x) + c$.

(iii) Evalution of Integrals of various Types :

Type I : Integrals of the form (i) $\int \frac{dx}{ax^2+bx+c}$ (ii) $\int \frac{dx}{\sqrt{(ax^2+bx+c)}}$ (iii) $\int \sqrt{(ax^2+bx+c)}\,dx$

Rule : Express $ax^2 + bx + c$ in the form of perfect square and then apply the standard resuts.

Type II : Integrals of the form

(i) $\int \frac{px+q}{ax^2+bx+c}$ (ii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$ (iii) $\int \frac{(p_0x^n+p_1x^{n-1}+......+p_{n-1}x+p_n)}{(ax^2+bx+c)}dx$

Rule for (I) : $\int \frac{(px+q)dx}{ax^2+bx+c} = \frac{b}{(2a)}\int \frac{(2ax+b)dx}{(ax^2+bx+c)} + \left(q-\frac{pb}{2a}\right)\int \frac{dx}{ax^2+bx+c}$

$$= \frac{p}{2a}\ln|ax^2+bx+c| + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

The other integral of R. H. S. can be evaluated with the help of type I.

Rule for (II) : $\int \frac{(px+q)dx}{\sqrt{ax^2+bx+c}} = \frac{p}{2a}\int \frac{(2ax+b)dx}{\sqrt{(ax^2+bx+c)}}dx + \left(q-\frac{pb}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$

$$= \frac{p}{a}\sqrt{(ax^2+bx+c)} + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

The other integral of R. H. S. can be evaluated with the help of type I.

Rule for (III) : In this case by actual division reduce the fraction to the form $f(x) + \frac{(px+q)dx}{ax^2+bx+c}$ and then integrate.

Type III : Integrals of the form (i) $\int \frac{dx}{a+b\sin^2 x}$ (ii) $\int \frac{dx}{a+b\cos^2 x}$ (iii) $\int \frac{dx}{a\sin^2 x+b\cos^2 x}$

(iv) $\int \frac{dx}{(a\sin x+b\cos x)^2}$ (v) $\int \frac{\phi(\tan x)dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x+d}$

where $\phi(\tan x)$ is a polynomial in tan x.

Rule : We shall always in such cases divide above and below by $\cos^2 x$; then put tan x = t i.e. $\sec^2 x\,dx = dt$ then the question shall reduce to the forms $\int \frac{dt}{(at^2+bt+c)}$ or $\int \frac{\varphi(t)dt}{(at^2+bt+c)}$.

Type IV : Integrals of the form (i) $\int \frac{dx}{a+b\sin x}$ (ii) $\int \frac{dx}{a+b\cos x}$

(iii) $\int \frac{dx}{a\sin x+b\cos x+c}$ (iv) $\int \frac{(p\cos x+q\sin x)}{(a\cos x+b\sin x)}dx$ (v) $\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}$

Rule for (i), (ii), and (iii):

write $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ The numberator shall become $\sec^2 x/2$ and the denominator will be a quadratic in tan x/2. Putting tan x/2 = t i.e. $\sec^2 x/2\,dx = 2dt$ the question shall reduce to the form $\int \frac{dt}{at^2+bt+c}$. **Rule for (iv) :** express the numberator as

$I(D^r) + m(\text{d.c. of } D^r)$

find l and m by comparing the coefficients of sin x and cos x and split the integral into sum of two integrals as $l\int dx + m\int \frac{\text{d.c.of } D^r}{D^r} = lx + m\,ln\,|D^r| + c$

Rule for (v) : Express the numerator as
$l(D^r) + m$ (d.c. of D^r) + n, find l, m, and n by comparing the coefficients of sin x, cos x and constant term and split the integral into sum of three integrals as

$l\int dx + m\int \frac{\text{d.c.of } D^r}{D^r}dx + n\int \frac{dx}{D^r} = lx + m\, ln\, |D^r| + n\int \frac{dx}{D^r}$ and to evaluate $\int \frac{dx}{D^r}$ proceed by the method to evaluate rule (i), (ii) and (iii).

Type V : Integrals of the form

(i) $\int \frac{(x^2+a^2)dx}{(x^4+kx^2+a^4)}$ (ii) $\int \frac{(x^2-a^2)dx}{(x^4+kx^2+a^4)}$

where k is a constant, + ve, –ve or zero.

Rule for (i) and (ii) : Divide above and below by x^2 then putting (i) $t = x - \frac{a^2}{x}$

i.e., $dt = \left(1+\frac{a^2}{x^2}\right)dx$ and $dt = \left(1-\frac{a^2}{x^2}\right)dx$

then the questions shall reduce to the form

$\int \frac{dt}{t^2+c^2}$ or $\int \frac{dt}{t^2-c^2}$

Remember :

(i) $\int \frac{x^2dx}{x^4+kx^2+a^4} = \frac{1}{2}\int \frac{(x^2+a^2)dx}{(x^4+kx^2+x^4)} + \frac{1}{2}\int \frac{(x^2-a^2)dx}{(x^4+kx^2+a^4)}$

(ii) $\int \frac{dx}{(x^4+kx^2+a^4)} = \frac{1}{2a^2}\int \frac{(x^2+a^2)dx}{(x^4+kx^2+x^4)} - \frac{1}{2a^2}\int \frac{(x^2-a^2)dx}{(x^4+kx^2+a^4)}$

(iii) $\int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$

Type VI : Integrals of the form

(i) $\int \frac{dx}{(Ax+B)\sqrt{(ax+b)}}$ (ii) $\frac{dx}{(Ax^2+Bx+C)\sqrt{(ax+b)}}$ (iii) $\int \frac{dx}{(Ax+B)\sqrt{(ax^2+bx+c)}}$

(iv) $\int \frac{dx}{(Ax^2+Bx+c)\sqrt{(ax^2+bx+c)}}$

Rule for (i) and (ii) : Put $ax + b = t^2$

Rule for (iii) : Put $Ax + B = \frac{1}{t}$

Rule for (iv) : Put $\frac{ax^2+bx+c}{Ax^2+Bx+C} = t^2$

(iv) Integration by partial fraction :

Form of the Rational Function	Form of the Partial Fraction
1. $\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2. $\frac{px+q}{(x-a)^2}$;	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$;	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4. $\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$;	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorised further.

In the above table, A, B and C are real numbers to be determined suitably.

5.4 PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^a f(x)dx = 0$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $(a < c < b)$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(vii) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$ and $\int_0^{2a} f(x)dx = 0$, if $f(2a-x) = -f(x)$

(viii) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ is even function, i.e., $f(-x) = -f(x)$.

5.5 INTEGRATION AS A LIMIT OF SUMS.

$$\int_a^b f(x)dx = \underset{h\to 0}{\text{Lt}}\, h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots\ldots\ldots\ldots + f(a+\overline{n-1}h)]$$

or $= (b-a)\underset{h\to\infty}{\text{Lt}}\frac{1}{n}[f(a) + f(a+h) + f(a+2h) + \ldots\ldots\ldots + f(a+\overline{n-1}h)]$, where $h = \frac{b-a}{n}$

The following results are used for evalating questions based on limit of sums.

(i) $1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + (n-1) = \sum(n-1) = \frac{(n-1)n}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + (n-1)^2 = \sum(n-1)^2 = \frac{(n-1)n(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + (n-1)^3 = \sum(n-1)^3 = \left[\frac{(n-1)n}{2}\right]^2$

(iv) $a + ar + \ldots\ldots\ldots + ar^{n-1} = \frac{a[r^n-1]}{r-1}$ $(r \neq 1)$.

SOLVED PROBLEMS

Ex.1 ***Find f(x), if***
(i) $f'(x) = (x-1)^3$ and $f(2) + f(-2) = 0$
(ii) $f'(x) = a(\cos x + \sin x)$, $f(0) = 9$ and
$f\left(\frac{\pi}{2}\right) = 15$

Sol. (i) Here, $f'(x) = (x-1)^3$

$\therefore$ $f(x) = \int (x-1)^3 dx = \frac{1}{4}(x-1)^4 + C$

$\therefore$ $f(2) = \frac{1}{4}(2-1)^4 + C = = \frac{1}{4} + C$ Given that $f(2) + f(-2) = 0$, we have

$\Rightarrow \frac{1}{4} + C + \frac{81}{4} + C = 0 \quad \Rightarrow \frac{82}{4} + 2C = 0$

$\Rightarrow 20.5 + 2C = 0 \quad \Rightarrow C = -10.25$

Hence, from (1) we have

$f(x) = \frac{1}{4}(x-1)^4 - 10.25 = \frac{1}{4}[(x-1)^4 - 41]$

(ii) Here, $f'(x) = a(\cos x + \sin x)$

$\therefore$ $f(x) = \int a(\cos x + \sin x)dx$

$= a\int \cos x\,dx + a\int \sin x\,dx$

$= a \sin x - a\cos x + C$ ...(1)

$\therefore f(0) = 9 = a\times 0 - a\times 1 + C$ ...(2) $\Rightarrow C - a = 9$

and $f\left(\frac{\pi}{2}\right) = 15 = a\times 1 - a\times 0 + C \Rightarrow a + C = 15$

From (2) and (3), we have a = 3, C = 12

Hence, from (1), we have $f(x) = 3(\sin x - \cos x) + 12$

Ex.2 ***Integrate the following functions w.r.t. x :***

(i) $\frac{(1+\log x)^2}{x}$ ***(ii)*** $\frac{(x+1)(x+\log x)^2}{x}$

(iii) $x\sqrt{x+2}$

Sol. (i) Put $1 + \log x = y$. Then, $\frac{1}{x}dx = dy$

So, $I = \int \frac{(1+\log x)^2}{x} dx$

$\Rightarrow \int y^2 dy = \frac{y^3}{3} + C = \frac{(1+\log x)^3}{3} + C$

(ii) Put $x + \log x = y$. Then, $\left(1 + \frac{1}{x}\right)dx = dy$

$\Rightarrow \left(\frac{x+1}{x}\right)dx = dy$

So, $I = \int \frac{(x+1)(x+\log x)^2}{x} dx$

$= \int y^2 dy = \frac{y^3}{3} + C = \frac{(x+\log x)^3}{3} + C$

(iii) Put $x + 2 = y$. Then, $dx = dy$

So, $I = \int x\sqrt{x+2}\,dy = \int (y-2)y^{\frac{1}{2}} dy$

$\Rightarrow \int (y^{\frac{3}{2}} - 2y^{\frac{1}{2}})\,dy = \int (y^{\frac{3}{2}}dy - 2)y^{\frac{1}{2}} dy$

$\Rightarrow \frac{2}{5}y^{\frac{5}{2}} - 2.\frac{2}{3}y^{\frac{2}{3}} + C = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$

Ex.3 ***Find :*** ***(i)*** $\int \cos^3 x\,dx$ ***(ii)*** $\int \sqrt{1-\sin x}\,dx$

Sol. (i) We have $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$\therefore \int \cos^3 x\,dx = \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x dx$

$= \frac{1}{4}\frac{\sin 3x}{3} + \frac{3}{4}\sin x + C = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C$

(ii) We have

$\sqrt{1-\sin x} = \sqrt{\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) - 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2} = \cos\frac{x}{2} - \sin\frac{x}{2}$

$\therefore \int \sqrt{1-\sin x}\,dx = \int \cos\frac{x}{2}dx - \int \sin\frac{x}{2}dx = 2\sin\frac{x}{2} + 2\cos\frac{x}{2} + C$

Ex.4 **Find : (i)** $\int \frac{x^2}{\sqrt{x^6+a^6}}dx$

(ii) $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$ **(iii)** $\int \frac{x-1}{\sqrt{x^2-1}}dx$

Sol. **(i)** Put $x^3 = t$ so that $3x^2\ dx = dt$ or $x^2\ dx = \frac{1}{3}dt$

$\therefore \int \frac{x^2}{\sqrt{x^6-a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+(a^3)^2}} = \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C = \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$

(ii) Put $\tan x = t$ so that $\sec^2 x\ dx = dt$

$\therefore \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+4}}$

$= \log\left|t+\sqrt{t^2+4}\right| + C = \log\left|\tan x+\sqrt{\tan^2 x+4}\right| + C$

(iii) $\int \frac{x-1}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx - \int \frac{dx}{\sqrt{x^2-1}}dx$

$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2+1}\right| + C$

Ex.5 **Find : (i)** $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$ **(ii)** $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Sol. (i) Let $x + 2 \equiv A.\frac{d}{dx}(x^2 + 2x + 3) + B$

$\Rightarrow x + 2 \equiv A.(2x + 2) + B$

On comparing coefficients of like powers of x, we have

$1 = 2A$ and $2 = 2A + B \Rightarrow A = \frac{1}{2}$ and $B = 1$

So, $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$

$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$

$= \frac{1}{2}\times 2\sqrt{x^2+2x+3} + C_1 + \int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$

$= \sqrt{x^2+2x+3} + C_1 + \log\left|x+1+\sqrt{x^2+2x+3}\right| + C_2$

$= \sqrt{x^2+2x+3} + \log\left|x+1+\sqrt{x^2+2x+3}\right| + C$

(ii) Let $5x + 3 \equiv A.\frac{d}{dx}(x^2 + 4x + 10) + B$

$\Rightarrow 5x + 3 \equiv A(2x + 4) + B$

On comparing coefficients of x, we have $5 = 2A$

and $3 = 4A + B \Rightarrow A = \frac{5}{2}$ and $B = -7$

$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$

$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$

$= \frac{5}{2}\times 2\sqrt{x^2+4x+10} + C_1 - 7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$

$= 5\sqrt{x^2+4x+10} + C_1 - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C_2$

$= 5\sqrt{x^2+4x+10} - 7\log\left|x+2+\sqrt{x^2+4x+10}\right| + C$

Ex.6 **Find :** $\int \frac{dx}{(x+1)(x+2)}$

Sol. We write $\frac{1}{(x+1)(x+2)} \equiv \frac{A}{x+1} - \frac{B}{x+2}$

where real numbers A and B are to be determined suitabley. This gives

$1 \equiv A(x + 2) + B(x + 1)$

Equation the coefficient of x and the constant terms, we get

$A + B = 0$ and $2A + B = 1 \Rightarrow A = 1$ and $B = -1$

Thus, $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$

$\Rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx$

$= \log|x+1| - \log|x+2| + C = \log\left|\frac{x-1}{x+2}\right| + C$

Ex.7 ***Find : (i)*** $\int \frac{3x-1}{(x+2)^2}dx$ ***(ii)*** $\int \frac{x}{(x-1)^2(x+2)}dx$

Sol. (i) Let $\frac{3x-1}{(x+2)^2} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2}$

Then, $3x - 1 \equiv A(x + 2) + B$

Comparing coefficients of x and the constant terms on both sides, we get

$3 = A$ and $-1 = 2A + B \Rightarrow A = 3$ and $B = -7$

Thus, $\int \frac{3x-1}{(x+2)^2}dx = \int \frac{3}{x+2}dx - \int \frac{7}{(x+2)^2}dx$

$= 3 \log |x + 2| - 7 \cdot \frac{1}{(x+2)} + C$

(ii) Let $\frac{x}{(x-1)^2(x+2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

Then $x = A(x-1)(x+2) + B(x+2) + C(x+1)^2$

$x = A(x^2 + x - 2) + B(x + 2) + C(x^2 - 2x + 1)$

Comparing coefficients of x^2, x and the constant terms on both sides, we get

$0 = A + C$; $1 = A + B - 2C$; $0 = -2A + 2B + C$

$\Rightarrow A = \frac{2}{9}$; $B = \frac{1}{3}$; $C = -\frac{2}{9}$

Thus, $\int \frac{x}{(x-1)^2(x+2)}dx$

$= \int \frac{\frac{2}{9}}{x-1}dx + \int \frac{\frac{1}{3}}{(x-1)^2}dx - \int \frac{\frac{2}{9}}{x+2}dx$

$= \frac{2}{9}\log|x-1| - \frac{1}{3(x-1)} - \frac{2}{9}\log|x+2| + C$

$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$

Ex.8 ***Find : (i)*** $\int x\sin^{-1}x\,dx$ ***(ii)*** $\int x\cos^{-1}x\,dx$

Sol. **(i)** Put $x = \sin t$ so that $dx = \cos t \,.\, dt$

So, $\int x\sin^{-1}x\,dx = \int \sin t \,.\, t \,.\, \cos t\,dt$

$= \frac{1}{2}\int t.\sin 2t\,dt = \frac{1}{2}\left[t.\int \sin 2t\,dt - \int\left[1.\int \sin 2t\,dt\right]dt\right]$

$= \frac{1}{2}\left[t.\frac{\cos 2t}{-2} + \int \frac{\cos 2t}{2}dt\right] = -\frac{1}{4}t.\cos 2t + \frac{1}{4}.\frac{\sin 2t}{2} + C$

$= \frac{1}{4}t.(1-2\sin^2 t) + \frac{1}{4}\sin t\sqrt{1-\sin^2 t} + C$

$= \frac{1}{4}(2x^2-1)\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + C$

(ii) Put $x = \cos t$ so that $dx = -\sin t\,dt$

So, $\int x\cos^{-1}x\,dx = -\int \cos t \,.\, t \,.\, \sin t\,dt$

$= -\frac{1}{2}\int t\sin 2t\,dt = \frac{1}{4}t.\cos 2t - \frac{1}{4}.\sin\frac{2t}{2} + C$

(Using (1) give below)

$= \frac{1}{4}t(2\cos^2 t - 1) - \frac{1}{4}\sqrt{1-\cos^2 t} \,.\, \cos t + C$

$= \frac{1}{4}(2x^2 - 1)\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + C$

Ex.9 ***Find : (i)*** $\int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dt$

(ii) $\int e^x\left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$

Sol. **(i)** Let $\log x = t$. Then $x = e^t$ and $\frac{dx}{x} = dt$

$\therefore \int\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx = \int\left(\frac{1}{t} - \frac{1}{t^2}\right)e^t\,dx$

$= \int e^t[f(t) + f'(t)]dt$, where $f(t) = \frac{1}{t}$ $= e^t \,.\, f(t) + C$

$= e^t \,.\, \frac{1}{t} + C = \frac{x}{\log x} + C$

(ii) $\int e^x\left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$

$= \int e^x\left[\frac{\sin 4x}{1-\cos 4x} - \frac{4}{1-\cos 4x}\right]dx$

$= \int e^x\left[\frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right]dx$

$= \int e^x\left[\cot 2x - 2\operatorname{cosec}^2 2x\right]dx$

$= \int e^x[f(x) + f'(x)]dx$, where $f(x) = \cot 2x$

$= e^x \,.\, f(x) + C = e^x \,.\, \cot 2x + C$

Ex.10 Find : (i) $\int \sqrt{x^2 - 8x + 7}\, dx$ **(ii)** $\int \sqrt{1 - 4x - x^2}\, dx$

Sol. (i) $I = \int \sqrt{x^2 - 8x + 7}\, dx = \int \sqrt{[(x-4)^2 - 9]}\, dx$

$$= \int \sqrt{(x-4)^2 - 3^2}\, dx = \frac{x-4}{2}\sqrt{x^2 - 8x + 7} - \frac{9}{2}$$

$$\log \left|(x-4) + \sqrt{x^2 - 8x + 7}\right| + C$$

(ii) $I = \int \sqrt{1 - 4x - x^2}\, dx = \int \sqrt{-(x^2 + 4x - 1)}\, dx$

$$= \int \sqrt{-[(x+2)^2 - 5]}\, dx = \int \sqrt{5 - (x+2)^2}\, dx$$

$$= \frac{1}{2}(x + 2)\sqrt{1 - 4x - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

Ex.11 Evaluate te following definite integrals as limit of sums : $\int_0^2 x\, dx$

Sol. We have $\int_a^b f(x) = \lim_{h \to 0} h\{f(a) + f(a + h) + f(a + 2h) + \ldots\ldots.. + f[a + (n - 1)h]\}$

where $h = \frac{b-a}{n}$

(i) Here, a = 0, b = 2, f(x) = x and

$$h = \frac{2-0}{n} = \frac{2}{n} \text{ or } nh = 2$$

$$\therefore \int_0^2 x\, dx = \lim_{h \to 0} h\{f(0) + f(0 + h) + f(0 + 2h) + \ldots\ldots. + f[0 + (n - 1)h]\}$$

$$= \lim_{h \to 0} h\{0 + (0+h) + (0+2h) + \ldots. + [0 + (n-1)h]\}$$

$$= \lim_{h \to 0} h[h + 2h + \ldots\ldots + (n - 1)h]$$

$$= \lim_{h \to 0} h^2[1+2+\ldots. + (n - 1)] = \lim_{h \to \infty} \left(\frac{2}{n}\right)^2 \left[\frac{(n-1)n}{2}\right]$$

$$(\because\ h = \frac{2}{n} \text{ and as } h \to 0,\ n \to \infty)$$

$$= \lim_{n \to \infty} \left[2\left(1 - \frac{1}{n}\right)\right] = 2$$

Ex.12 Show that $\int_0^{\pi/2} f(\sin 2x) \sin x\, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x\, dx.$

Sol. Let $x = \frac{\pi}{4} - t$. Then $t = \frac{\pi}{4} - x \Rightarrow dt = -dx$

When x = 0, $t = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ and when $x = \frac{\pi}{2}$,

$t = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$ $\quad \therefore \int_0^{\pi/2} f(\sin 2x) \sin x\, dx$

$$= \int_{\pi/4}^{-\pi/4} f\left\{\sin 2\left(\frac{\pi}{4} - t\right)\right\} \sin\left(\frac{\pi}{4} - t\right)(-dt)$$

$$= \int_{-\pi/4}^{\pi/4} f(\cos 2t)\left(\sin\frac{\pi}{4}\cos t - \cos\frac{\pi}{4}\sin t\right) dt$$

$$= \int_{-\pi/4}^{\pi/4} f(\cos 2t)\left(\frac{1}{\sqrt{2}}\cos t - \frac{1}{\sqrt{2}}\sin t\right) dt$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\cos t\, dt - \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\sin t\, dt$$

$$= \frac{1}{\sqrt{2}}.2\int_0^{\pi/4} f(\cos 2t)\cos t.dt - \frac{1}{\sqrt{2}}.0$$

[$\because$ f (cos 2t) cos t is an even function ; and f (cos 2t) sin t is an odd function]

$$= -\sqrt{2}\int_0^{\pi/4} f(\cos 2t)\cos t\, dt$$

EXERCISE – I UNSOLVED PROBLEMS

Q.1. Evalute : (i) $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x}dx$ (ii) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx$ (iii) $\int \sqrt{1+\cos 3x}dx$

Q.2 Evalute : (i) $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$ (ii) $\int \tan^{-1}(\sec x + \tan x)dx$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Q.3 (i) If f′(x) = sin2x + cos3x + 5, f(0) = $\frac{3}{2}$, find (x)

(ii) If f′(x) = a sinx + b cos x and f′(0) = 4, f(0) = 3, $f\left(\frac{\pi}{2}\right)$ = 5, find f(x)

(iii) The gradient of a curve is given by $= 2x - \frac{3}{x^2}$. The curve passes through (1,1). Find its equation.

(iv) The gradient of a curve is $6x^2 - 2ax + a^2$. The curve passes through the point (0, 0) and (1, 8). Find a.

Q.4 Evaluate : (i) $\int \sin x \sin 2x \sin 3x \, dx$ (ii) $\int \tan x \tan 2x \tan 3x \, dx$

Q.5 Evaluate : (i) $\int \frac{\sin(x-a)}{\sin x}dx$ (ii) $\int \frac{\sin x}{\sin(x-a)}dx$ (iii) $\int \frac{1}{\sin(x-a)\sin(x-b)}dx$

Q.6 Evalute : (i) $\int \frac{1}{e^x + e^{-x}}dx$ (ii) $\int \frac{1}{x^{1/2} + x^{1/3}}dx$

Q.7 Evaluate : (i) $\int \frac{\sin 2x}{(a+b\cos x)^2}dx$ (ii) $\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}dx$, $a \neq n\pi$, $n \in z$

Q.8 Evalute : (i) $\int \log(1+x)^{(1+x)}dx$ (ii) $\int \frac{\sin^{-1} x}{x^2}dx$

Q.9 Evaluate : (i) $\int \sec^3 x \, dx$ (ii) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

Q.10 Evaluate : (i) $\int \sin(\log x)dx$ (ii) $\int e^{ax}\cos bx \, dx$

Q.11 Evaluate : $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Q.12 Evaluate : (i) $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}dx$ (ii) $\int \sqrt{\frac{a-x}{a+x}}dx$

Q.13 Evaluate : (i) $\int (3x-2)\sqrt{x^2+x+1}\,dx$ (ii) $\int (2x-5)\sqrt{x^2-4x+3}\,dx$

Q.14 Evaluate : (i) $\int \frac{1}{3+4\cos x}dx$ (ii) $\int \frac{\cos x}{1+\cos x+\sin x}dx$ (iii) $\int \frac{1}{3+2\sin x+\cos x}dx$

Q.15 Evaluate : (i) $\int \frac{1}{1+\cot x}dx$ (ii) $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$

Q.16 Evaluate : (i) $\int \frac{1}{(x^2+1)(x^2+4)}dx$ (ii) $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ (iii) $\int \frac{x^2+2}{(x^2+1)(x^2+4)}dx$

Q.17 Evaluate : (i) $\int \frac{1}{x^4+1}dx$ (ii) $\int \frac{x^2+1}{x^4+x^2+1}dx$ (iii) $\int (\sqrt{\tan x}+\sqrt{\cot x})dx$

Q.18 Evaluate : (i) $\int_e^{e^2}\left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx$ (ii) $\int_0^1\sqrt{\frac{1-x}{1+x}}dx$ (iii) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ (iv) $\int_0^{\pi/2}\frac{1}{2\cos x+4\sin x}dx$

Q.19 Evaluate as limit of sums : (i) $\int_a^b \sin x\,dx$ (ii) $\int_0^4 (x+e^{2x})dx$ (iii) $\int_1^3 (2x+3)dx$ (iv) $\int_1^3 (3x^2+2x+1)dx$

Q.20 Prove that $\int_0^{\pi/2}\log(\sin x)dx = \int_0^{\pi/2}\log(\cos x)dx = -\frac{\pi}{2}(\log 2)$

Q.21 Evaluate : (i) $\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$ (ii) $\int_{-\pi/2}^{\pi/2}\log\left|\frac{2-\sin x}{2+\sin x}\right|dx$

Q.22 Evaluate : $\int_{\pi/6}^{\pi/3}\frac{1}{1+\tan x}dx$

Q.23 Evaluate : (i) $\int_{-\pi/2}^{\pi/2}(\sin|x|+\cos|x|)dx$ (ii) $\int_{-1}^{1}e^{|x|}dx$ (iii) $\int_0^2 |x^2+2x-3|dx$

(iv) $\int_1^4 f(x)dx$, where $f(x) = |x-1| + |x-2| + |x-3|$

Q.24 Evaluate : (i) $\int_1^4 f(x)dx$, where $f(x) = \begin{cases} 3x^2+4, & \text{when } 0 \le x \le 2 \\ 9x-2, & \text{when } 2 \le x < 4 \end{cases}$

(ii) $\int_0^1 |5x-3|dx$ (iii) $\int_0^\pi |\cos x|dx$ (iv) $\int_0^{2\pi}|\sin x|dx$

Q.25 Evaluate : (i) $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$ (ii) $\int_0^\pi \frac{x}{\sin x+\cos x}dx$ (iii) $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

(iv) $\int_0^{\pi/2}\frac{x}{\sin x+\cos x}dx$ (v) $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

Q.26 Evaluate : (i) $\int_0^{\pi/2}\frac{\sin^3 x}{\sin^3 x+\cos^3 x}dx$ (ii) $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ (iii) $\int_0^{\pi/4}\log(1+\tan x)dx$

(iv) $\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ (v) $\int_0^1 x(1-x)^n\,dx$ (vi) $\int_0^{\pi/2}\sin 2x\,\log(\tan x)dx$

EXERCISE – II BOARD PROBLEMS

Q.1 (i) Evaluate $\int_0^{\pi/2} \frac{\sin^7 x}{\sin^7 x + \cos^7 x} dx$. (ii) $\int_0^3 (x^2 + 2x)\, dx$ as limit of a sum.

Q.2 (i) $\int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$ (ii) $\int \sin^4 x\, dx$. (iii) $\int \frac{1+\sin 2x}{x+\cos^2 x} dx$ (iv) $\int x \sin^{-1} x\, dx$.

Q.3 (i) $\int \sin 7x \sin x\, dx$. (ii) $\int \frac{4x+3}{\sqrt{2x^2+2x-3}} dx$ (iii) $\int \log(1+x^2)\, dx$. (iv) $\int \frac{x-1}{(x+1)(x-2)} dx$.

Q.4 (i) Evaluate as the limit of a sum $\int_0^2 (x^2+3)\, dx$. (ii) Evaluate $\int_0^{\pi} \frac{x}{1+\sin x} dx$. (iii) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$.

Q.5 (i) $\int e^{ax} \cos bx\, dx$. (ii) $\int \frac{x}{(x^2+1)(x+1)} dx$ (iii) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$ (iv) $\int \frac{dx}{x^2-4x+8}$.

Q.6 (i) Evaluate $\int_0^{\pi/2} \cos^2 x\, dx$. (ii) Prove that $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2}\pi$.

(iii) Prove $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8} \log 2$. (iv) Evaluate $\int_1^2 e^x \left[\frac{1}{x} - \frac{1}{x^2}\right] dx$.

Q.7 (i) $\int \sec^3 x$ (ii) $\int \frac{\tan^{-1} x}{(1+x)^2} dx$. (iii) $\int \tan^{-1} \sqrt{\frac{1-\sin x}{1+\sin x}}\, dx$.

(iv) $\int \frac{x\, dx}{(x+2)(3-2x)}$.

Q.8 (i) $\int_0^{\pi/2} \sin 2x . \log(\tan x)\, dx = 0$. (ii) Prove that $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx = a\pi$.

(iii) Evaluate $\int_{\pi/4}^{\pi/2} \cos 2x . \log \sin x\, dx$. (iv) $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

Q.9 (i) $\int (\sin^{-1} x)^2\, dx$. (ii) $\int \sin^4 2x\, dx$. (iii) $\int \frac{dx}{3+2\sin x+\cos x}$ (iv) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$. **[C.B.S.E. 2004]**

Q.10 (i) Evaluate $\int_{-5}^{0} f(x)\,dx$, where f(x) = |x| + |x + 3| + |x + 6|.

(ii) $\int_{0}^{2} (x^2 + x)\,dx$ as a limit of sums (iii) $\int_{0}^{\pi} \frac{dx}{5 + 4\cos x}$

Q.11 (i) Prove that $\int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\,dx = \frac{\pi}{2} - 1$. (ii) $\int_{0}^{\pi/4} \sqrt{1 - \sin 2x}\,dx = \sqrt{2} - 1$.

(iii) Evaluate $\int_{0}^{\pi/4} 2\tan^3 x\,dx$. (iv) $\int_{0}^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

Q.12 (i) $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}\,dx$. (ii) $\int \frac{\sqrt{16 + (\log x)^2}}{x}\,dx$. (iii) $\int \frac{2x - 1}{(x-1)(x+2)(x-3)}\,dx$.

(iv) $\int \frac{x^2}{x^2 + 6x + 12}\,dx$ (v) $\int \frac{\sin 2x}{(a + b\cos x)^2}\,dx$ (vi) $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$.

Q.13 (i) $\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx$. (ii) $\int (x+3)\sqrt{3 - 4x - x^2}\,dx$. (iii) $\int \frac{2x+1}{2x^2 + 4x - 3}\,dx$

(iv) $\int \sqrt{\tan\theta}\,d\theta$ (v) $\int \cos^4 x\,dx$ (vi) $\int \frac{x^2 + x + 1}{(x+2)(x+1)^2}\,dx$.

Q.14 (i) Evaluate $\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}}\,dx$. (ii) Prove that $\int_{0}^{a} f(x)\,dx = \int_{a}^{b} f(a - x)\,dx$. Hence evaluate $\int_{0}^{\pi/2} \frac{dx}{1 + \tan x}$.

Q.15 (i) $\int \sin 4x \sin 8x\,dx$ (ii) $\int \frac{\sin 2x}{(1 - \cos 2x)(2 - \cos 2x)}\,dx$ (iii) $\int \frac{1 - x^2}{1 + x^4}\,dx$

Q.16 (i) $\int_{0}^{3} (2x^2 + 3x + 5)\,dx$ as limit of sums (ii) $\int_{0}^{\pi} \frac{x\tan x}{\sec x\ \mathrm{cosec}\ x}\,dx$ (iii) $\int_{0}^{2} x\sqrt{2 - x}\,dx$ as limit of sums

Q.17 (i) $\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\,dx$ (ii) If $\int (e^{ax} + bx)\,dx = \frac{e^{4x}}{4} + \frac{3x^2}{2}$ find the value of a and b

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Q.18 (i) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ (ii) $\int \frac{x \sin x}{1+\cos^2 x} dx$ (iii) $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ (iv) $\int_0^{\pi} \cot^{-1}(1-x+x^2)\, dx$

Q.19 (i) $\int \frac{\sec^2 x}{3+\tan x} dx$ (ii) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$ (iii) $\int \frac{(x-4)\, e^x}{(x-2)^3} dx$ (iv) $\int x \sin^{-1} x\, dx$

Q.20 (i) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}} dx$ (ii) $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$ (iii) $\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Q.21 (i) $\int \sec^2(7-4x)\, dx$ (ii) $\int \frac{x+2}{\sqrt{(x-2)(x-3)}} dx$

Q.22 (i) $\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$ (ii) $\int_1^2 \frac{5x^2}{x^2+4x+3} dx$

Q.23 $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx$

Q.24 Evaluate : (i) $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx$ (ii) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Q.25 Evaluate (i) $\int_{-1}^{2} |x^3 - x|\, dx$. (ii) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Q.26 Evaluate (i) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ (ii) $\frac{x^2+1}{(x-1)^2(x+3)} dx$

Q.27 Evaluate :

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$$

Evaluate :

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

Q.28 Evaluate :

$$\int \frac{dx}{x(x^5+3)}$$

Q.29 Evaluate :

$$\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$$

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Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

1. (i) $-\cot x - \operatorname{cosec} x + c$ (ii) $\tan x - \cot x - 3x + c$ (iii) $\frac{2\sqrt{2}}{3} \sin \frac{3x}{2} + c$ **2.** (i) $\frac{x^2}{2} + c$ (ii) $\frac{\pi x}{4} + \frac{x^2}{4} + c$

3. (i) $-\frac{\cos 2x}{2} + \frac{\sin 2x}{3} + 5x + 2$ (ii) $2 \cos x + 4 \sin x + 1$ (iii) $y = x^2 + \frac{3}{x} - 3$ (iv) 3, –2

4. (i) $-\frac{\cos 4x}{16} - \frac{\cos 2x}{8} + \frac{\cos 6x}{24} + c$ (ii) $-\frac{1}{3} \log \cos 3x + \frac{1}{2} \log \cos 2x + \frac{1}{2} \log \cos x + c$

5. (i) $x \cos a - \sin a \log \sin x + c$ (ii) $\sin a \log \sin (x - a) + (x - a) \cos a$ (iii) $\operatorname{cosec} (a - b) \log \frac{\sin(x-a)}{\sin(x-b)} + c$

6. (i) $\tan^{-1} x\, e + c$ (ii) $2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log (x^{1/6} + 1) + c$ **7.** (i) $\frac{-2}{b^2}\left[\log(a + b\cos x) + \frac{a}{a + b\cos x}\right] + c$

(ii) $-2 \operatorname{cosec} a \sqrt{\cos a + \cot x + \sin a}$ **8.** (i) $\frac{(x+1)^2}{x^2} \log (x + 1) - \frac{x^2}{4} - \frac{x}{2} + c$ (ii) $-\frac{\sin^{-1} x}{x} + \log\left[\frac{1-\sqrt{1-x}}{x}\right]^2$

9. (i) $\frac{\sec x \times \tan x}{2} + \frac{1}{2} \log \tan\left(\frac{\pi}{2} + \frac{x}{2}\right) + c$ (ii) $\frac{2}{\pi}(2x - 1) \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x - x^2} - x + c$

10. (i) $\frac{x}{2}[\sin(\log x) - \cos (\log x)] + c$ (ii) $\frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] + c$

11. (i) $-2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$

12. (i) $-\cos \alpha \sin^{-1}\left(\frac{\cos x}{\cos \alpha}\right) - \sin \alpha \log [\sin x + \sqrt{\sin^2 x - \sin^2 \alpha}\,] + c$ (ii) $a \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + c$

13. (i) $(x^2 + x + 1)^{3/2} - \frac{7(2x+1)}{8}\sqrt{x^2 + x + 1} - \frac{21}{16} \log \left[\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1}\right] + c$

(ii) $\frac{1}{3}(x^2 - 4x + 3)^{3/2} - \frac{1}{2}(x - 2) \sqrt{x^2 - 4x + 3} - \frac{1}{2} \log [(x - 2) + \sqrt{x^2 - 4x + 3}\,]$

14. (i) $\frac{1}{\sqrt{7}} \log \left[\frac{3\tan\frac{x}{2} + 4 - \sqrt{7}}{3\tan\frac{x}{2} + 4 + \sqrt{7}}\right] + c$ (ii) $\frac{x}{2} + \frac{1}{2}\log\left[\frac{1 + \cos x + \sin x}{1 + \tan\frac{x}{2}}\right] + c$ (iii) $\tan^{-1}\left(\tan\frac{x}{2} + 1\right) + c$

15. (i) $\frac{x}{2} - \frac{1}{2} \log (\sin x + \cos x) + c$ (ii) $\frac{12x}{13} - \frac{5}{13}\log (3 \cos x + 2 \sin x) + c$

16. (i) $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right)$ (ii) $\frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right)$ + c **17.** (i) $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left[\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}x+1}\right]+c$

(ii) $\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{x^2-1}{\sqrt{3}\,x}\right]+c$ (iii) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+c$

18. (i) $\frac{e^2}{2}-e$ (ii) $\frac{\pi}{2}-1$ (iii) $\frac{\pi}{2ab(a+b)}$ (iv) $\frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{5}+1}{2(\sqrt{5}-2)}\right]$

19. (i) cos a−cos b (ii) $\frac{15+e^8}{2}$ (iii) 14 (iv) 36 **21.** (i) $\frac{\pi}{4}-\frac{1}{2}$ (ii) 0 **22.** $\frac{\pi}{12}$

23. (i) 4 (ii) 2e−2 (iii) 4 (iv) $\frac{19}{2}$ **24.** (i) 66 (ii) $\frac{13}{10}$ (iii) 2 (iv) 4 **25.** (i) $-\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$ (ii) π

(iii) $\pi\left(\frac{\pi}{2}-1\right)$ (iv) $\frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ (v) $\frac{\pi^2}{4}$ **26.** (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{8}\log 2$ (iv) 0 (v) $\frac{1}{(n+1)(n+2)}$ (vi) 0

EXERCISE – 2 (BOARD PROBLEMS)

1. (i) $\frac{\pi}{4}$ (ii) 18 **2.** (i) $2\sqrt{x^2+4x+3}-3\log\left[(x+2)+\sqrt{x^2+4x+3}\right]+c$ (ii) $\frac{1}{32}$ [12x − 8 sin 2x+sin 4x]+c

(iii) log (x + $\cos^2$x)+c (iv) $\frac{x^2}{2}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{4}\sin^{-1}x + c$ **13.** (i) $\pi/2$ (ii) $\pi/12$

16. (i) $\pi\left(\frac{\pi}{2}-1\right)$ (ii) $\frac{\pi^2}{4}$ (iii) $\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$ (iv) $\frac{\pi}{2}-\log 2$ **17.** (i) $\frac{93}{2}$ (ii) $\frac{\pi^2}{4}$ (iii) $\frac{16}{15}\sqrt{2}$ **18.** (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{4}$

(iii) $\frac{3}{5\sqrt{2}}$ (iv) $\frac{20}{3}$ **3.** (i) $\frac{\sin 6x}{12}-\frac{\sin 8x}{16}$ + c (ii) $2\sqrt{2x^2+2x-3}+\frac{1}{\sqrt{2}}\log\left[x+\frac{1}{2}+\sqrt{x^2+x-\frac{3}{2}}\right]$

(iii) x log (1 + x^2) − 2(x − $\tan^{-1}$x) + c (iv) $\frac{2}{3}\log(x+1) + \frac{1}{3}\log(x-2) + c$ **4.** (i) $\frac{26}{3}$ (ii) π (iii) $\frac{\pi^2}{4}$

5. (i) $\frac{e^{ax}}{a^2+b^2}$ [a cos bx + b sin bx] + c (ii) $\frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\log(x+1)$

(iii) $\frac{2}{3}\log(x+2)^{3/2} - \frac{2}{3}\log(x+1)^{3/2} + c$ (iv) $\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)$ **6.** (i) $\frac{\pi}{4}$ (iv) $\frac{e^2}{2}-e$

7. (i) sec x tan x + log (secx + tan x) (ii) $\frac{\tan^{-1}x}{1+x} + \frac{1}{2}$ [log (1 + x) + $\tan^{-1}x - \frac{1}{2}\log(1 + x^2)$] + c (iii) $\frac{\pi x}{4} - \frac{x^2}{4} + c$

(iv) $-\frac{2}{7}\log(x + 2) - \frac{3}{14}(3 - 2x) + c$ **8.** (iii) $\frac{1}{4}\left(1-\frac{\pi}{2}+\log 2\right)$ (iv) $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

9. (i) $x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + c$ (ii) $\frac{1}{8}[3x - \sin 4x + \frac{1}{8}\sin 8x] + c$ (iii) $\tan^{-1}\left(1+\tan\frac{x}{2}\right) + c$

(iv) $\sin^{-1}\left(\frac{e^x+2}{3}\right) + c$ **10.** (i) $\frac{73}{2}$ (ii) $\frac{14}{3}$ (iii) $\frac{\pi}{3}$ **11.** (iii) 1−log 2 (iv) $\log\frac{4}{3}$

12. (i) $\frac{1}{a^2-b^2}\log[a^2\sin^2x + b^2\cos^2x] + c$ (ii) $\frac{\log x\sqrt{16+(\log x)^2}}{2} + \log[\log x + \sqrt{16+(\log x)^2}]$

(iii) $-\frac{1}{6}\log(x - 1) - \frac{1}{3}\log(x + 2) + \frac{1}{2}\log(x - 3)$ (iv) $x - 3\log(x^2 + 6x + 12) + \frac{6}{\sqrt{3}} + \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + c$

(v) $-\frac{2}{b^2}\left[\log(a+b\cos x)+\frac{a}{a+b\cos x}\right] + c$ (vi) $\frac{x}{\log x} + c$ **13.** (i) $e^x\left(\frac{x-1}{x+1}\right) + c$

(ii) $-\frac{1}{3}[3 - 4x - x^2]^{3/2} + \frac{(x+2)}{2}\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + c$

(iii) $\frac{1}{2}\log[2x^2 + 4x - 3] - \frac{\sqrt{2}}{4\sqrt{5}}\log\left[\frac{\sqrt{2}(x+1)-\sqrt{5}}{\sqrt{2}(x+1)+\sqrt{5}}\right]$ (iv) $\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right] + \frac{1}{2\sqrt{2}}\left[\frac{\tan^2\theta-\sqrt{2}\tan\theta+1}{\tan^2\theta+\sqrt{2}\tan\theta+1}\right] + c$

(v) $\frac{3x}{8} + \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + c$ (vi) $-2\log(x + 1) - \frac{1}{x+1} + 3\log(x + 2) + c$

15. (i) $\frac{1}{24}(3\sin x - \sin 12x) + c$ (ii) $\frac{1}{2}\log\left(\frac{1-\cos 2x}{2-\cos 2x}\right) + c$ (iii) $\frac{-1}{2\sqrt{2}}\log\left[\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right] + c$

17. (i) $\frac{1}{3}\log(x^3 + 1) + c$ (ii) a = 4, b = 3 **23.** $6.\sqrt{x^2-9x+20} + 34.\log\left|\left(\frac{2x-9}{2}\right)+\sqrt{x^2-9x+20}\right| + c$

25. (i) $\frac{11}{4}$ (ii) $\frac{\pi^2}{4}$

26. (i) $-\sin^{-1}x\left(\sqrt{1-x^2}\right) + x + c$ (ii) $\frac{3}{8}\log|x-1| + \frac{5}{8}\log|x+3| - \frac{1}{2}.\frac{1}{(x-1)} + C$