SQUARE AND SQUARE ROOTS

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SQUARES

When a number is multiplied by itself the product is called the square of the number.

for eg., $2 \times 2 = 4$ or $2^2 = 4$. We say that the square of 2 is 4. Similarly, $3 \times 3 = 9$ or $3^2 = 9$, etc.

Definition :- A natural number is said to be a perfect square, if it is the square of another natural number.

For Ex., $5 \times 5 = 5^2 = 25$, $6 \times 6 = 6^2 = 36$, $7 \times 7 = 7^2 = 49$, etc.

So 4, 9, 16, 25, 36,……. are all perfect squares.

The squares of the first 30 natural numbers are :

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
$4^2 = 16$	$5^2 = 25$	$6^2 = 36$
$7^2 = 49$	$8^2 = 64$	$9^2 = 81$
$10^2 = 100$	$11^2 = 121$	$12^2 = 144$
$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$
$19^2 = 361$	$20^2 = 400$	$21^2 = 441$
$22^2 = 484$	$23^2 = 529$	$24^2 = 576$
$25^2 = 625$	$26^2 = 676$	$27^2 = 729$
$28^2 = 784$	$29^2 = 841$	$30^2 = 900$

Note : For a number to be a perfect square, it should not have 2, 3, 7 or 8 in its units place.
Also, a number is not a perfect square if it ends with odd number of zeros. It should have even number of zeros at the end.

Eg. $9^2 = 9 \times 9 = 81$; $13^2 = 13 \times 13 = 169$
$16^2 = 16 \times 16 = 256$; $17^2 = 17 \times 17 = 289$
$20^2 = 20 \times 20 = 400$ $21^2 = 21 \times 21 = 441$
$35^2 = 35 \times 35 = 1225$ $90^2 = 90 \times 90 = 8100$
$100^2 = 100 \times 100 = 10000$

Tip : $15^2 = (1 \times 2)25 = 225$, $25^2 = (2 \times 3)25 = 625$ etc

SQUARE ROOTS

Since

$2^2 = 4$, or square root of 4 is ±2.

$3^2 = 9$, or square root of 9 is ±3.

$9^2 = 81$, or square root of 81 is ±9.

So the square root of a number x is that number which when multiplied by itself gives the number x itself. The number x under consideration is a perfect square.

The symbol of square root is $\sqrt{x}$ or $(x)^{1/2}$

Thus, the facts such as the square root of 4 is 2, of 81 is 9, etc., can be mathematically represented as :

$\sqrt{4} = 2$ and -2, $\sqrt{81} = 9$ and -9

Note :

(1) The square of a number is always positive.
(2) Square root of a number is + and –
(3) In this class, we consider only positive square root

Square Roots of Perfect Squares

There are two methods for finding out the square root of a number,

1. Prime Factorisation Method
2. Long Division Method

Prime Factorisation Method

Step 1. Find the prime factors of the number.

Step 2. Then, pair the factors and choose one prime from each pair.

Step 3. Find the product of all such primes so taken. This gives the square root of the given number.

Ex.1 Find the square root of 441.

Sol. $441 = 3 \times 3 \times 7 \times 7$

$\sqrt{441} = \sqrt{3\times3\times7\times7}$

$= 3 \times 7 = 21$

Verification : $21^2 = 21 \times 21 = 441$.

3	441
3	147
7	49
	7

Ex.2 Find the square root of 1764.

Sol. $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$

Taking out one factor from every pair,

$\sqrt{1764} = \sqrt{2\times2\times3\times3\times7\times7}$

$= 2 \times 3 \times 7 = 42$

2	1764
2	882
3	441
3	147
7	49
	7

Ex.3 Is 4050 a perfect square ? If not then find the smallest number by which it should be multiplied to make it a perfect square. Also, find the square root.

Sol. Prime factorise are

$4050 = 2 \times 5 \times 5 \times 3 \times 3 \times 3 \times 3$

$= 2 \times 5^2 \times 3^2 \times 3^2$

No, 4050 is not a perfect square

Now, to make 4050 a perfect square, it should be multiplied by 2, so that the pairing can be complete.

Thus, the new number is $4050 \times 2 = 8100$.

Now, the square root of 8100 is given by $\sqrt{8100}$.

$\sqrt{8100} = \sqrt{2\times2\times5\times5\times3\times3\times3\times3}$

$= 2 \times 5 \times 3 \times 3 = 90$

2	4050
5	2025
5	405
3	81
3	27
3	9
3	3
	1

Ex.4 Is 9408 a perfect square ? Find the smallest number by which it should be divided to make it a perfect square. Also find the square root.

Sol. First find the prime factorisation of 9408.

$9408 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 7 \times 7 \times 3$

Now, to make 9408 a perfect square, we divide 9408 by 3.

Then the new number is 3136.

$\therefore \sqrt{3136} = \sqrt{2\times2\times2\times2\times2\times2\times7\times7}$

$= 2 \times 2 \times 2 \times 7 = 56.$

Ex.5 Find the smallest number by which 9408 must be divided so that the quotient is a perfect square. Find the square root of the quotient.

Sol. We have,

$9408 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times 3 \times \underline{7 \times 7}$.

If we divide 9408 by the factor 3, then

$9408 \div 3 = 3136$

$= \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{7 \times 7}$

Which is a perfect square. Therefore, the required smallest number is 3.

And, $\sqrt{3136} = 2 \times 2 \times 2 \times 7 = 56$

Ex. 6. Find the smallest square number which is divisible by each of the numbers 6,9 and 15.

Sol. This has to be done in two steps. First find the smallest common multiple and then find the square number needed. The least number divisible by each one of 6,9 and 15 is their LCM. The LCM of 6,9 and 15 is

$2 \times 3 \times 3 \times 5 = 90$

2	6,9,15
3	3,9,15
3	1,3,5
5	1,1,5
	1,1,1

Prime factorisation of 90 is

$90 = 2 \times \underline{3 \times 3} \times 5$

We see that prime factors 2 and 5 are not in pairs. Therefore 90 is not a perfect square.

In order to get a perfect square, each factor of 90 must be paired. So we need to make pairs of 2 and 5. Therefore, 90 should be multiplied by 2×5, i.e., 10.

Hence, the required square number is

$90 \times 10 = 900$.

➤ LONG DIVISION METHOD

Step 1. The digits of the number are paired-off, starting from the units place. Each pair is called a period.

Step 2. Find a digit whose square is less than or equal to the left most period which is the first dividend. This digit is the divisor as well as the quotient.

Step 3. Subtract the product of the divisor and the quotient from the first period. Bring down the next period to the right of the remainder. This is the new dividend.

Step 4. Now double the quotient to get the new divisor with a blank space on the right and assign a largest possible digit both in the quotient and in the blank space, such that the product of this digit with the new divisor having the same digit in the blank space is equal to or less than the dividend.

Repeat steps 3 and 4 till all the periods are over. The quotient thus obtained is the square root of the given number.

Ex. 7. Find the square root of 5476 by using long division method.

Sol.

```
        74
    7 | 54 76
      | -49
  144 |  5 76
      | -5 76
      |     0
```

Step 1. Pair the digits starting from the units digit.

Step 2. Find a whole number whose square is just equal to or less than 54, i.e. 7 is the divisor and the quotient.

Step 3. Subtract the product ($7 \times 7 = 49$) from 54.

Step 4. Double the quotient ($7 \times 2 = 14$) and bring down the next pair, i.e. 76 the new dividend.

Step 5. Now think of a digit to be placed on the right of 14, such that the digit multiplied by the divisor will give 576.
Since $4 \times 14\underline{4} = 576$; the square root is 74.
$\therefore \sqrt{5476} = 74$

Ex.8 Find the square root of 7744 by using long division method.

Sol.

```
        88
    8 | 77 44
      | -64
  168 | 13 44
      |-13 44
      |     0
```

Step 1. $8 \times 8 = 64 < 77$
Step 2. $77 - 64 = 13$
Step 3. Double of 8 is 16 and bring down the next period, i.e. 44.
Step 4. Since $16\underline{8} \times 8 = 1344$, 8 is the next digit of the quotient.
$\therefore \sqrt{7744} = 88$

Ex.9 Find the square root of 529 by using long division method.

Sol. Observe that 5 does not have a pair, so
Step 1. $2 \times 2 = 4$ and $5 - 4 = 1$
Now bring down 29.
Step 2. $43 \times 3 = 129$

```
       23
    2 | 5 29
      | -4
   43 | 1 29
      |-1 29
      |    0
```

$\therefore \sqrt{529} = 23$

Ex.10 What must be added to 7581 to make it a perfect square ?

Sol. First finding the square root of 7581.
Now, 87^2 is 12 less than 7581, so we choose the square of 88.
Thus, $7581 < 88^2$.
Now, $88^2 = 88 \times 88 = 7744$

```
        87
    8 | 75 81
      | -64
  167 | 11 81
      |-11 69
      |    12
```

$\therefore$ The least number to be added
$= 7744 - 7581 = 163$
So, 163 should be added to 7581 to make it a perfect square.

Ex.11 Find what is the least number that should be added to 2361 to make it a perfect square ?

Sol. Now, 48^2 is 57 less than 2361, so we choose the square of 49.

Thus, $2361 < 49^2$.

Now, $49^2 = 49 \times 49 = 2401$

```
        48
     ______
  4 | 2361
    | -16
 88 |  761
    | -704
    ------
        57
```

The number to be added is 2401 – 2361 = 40

Ex.12 Find the biggest 4-digit number which is a perfect square.

Sol. The biggest 4-digit number is 9999.

Now $99^2 < 9999$, while $100^2 = 100 \times 100$

$= 10000$

$\therefore 99^2 = 99 \times 99 = 9801$

$\therefore$ 9801 is the largest 4-digit number which is a perfect square

```
         99
      ______
   9 | 9999
     | -81
 189 | 1899
     |-1701
     ------
        198
```

Number of digits in the square root of a number

Put a dot on the units digit of a number. Now put a dot over every alternate digit. The number of dots gives the number of digits in the square root of the given number. For Ex.

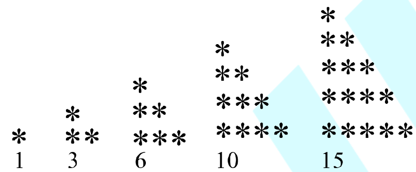
Number	Number of digits in the square root	Square root
$\dot{1}1\dot{4}4\dot{9}$	3	107
$3\dot{4}8\dot{1}$	2	59
$\dot{1}2\dot{6}3\dot{3}7\dot{6}$	4	1124

Note: The square root of a two or one-digit number has one digit, while a three-digit number will have a square root with two digits.

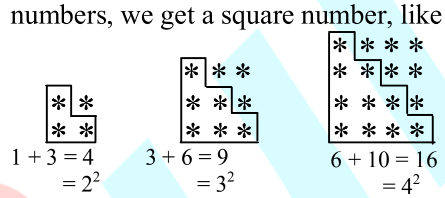
➤ SOME MORE INTERESTING PATTERNS

1. Adding triangular numbers.

Triangular Numbers : Numbers whose dot patterns can be arranged as triangles.



If we combine two consecutive triangular numbers, we get a square number, like



2. Numbers between square numbers

We can find some interesting pattern between two consecutive square numbers.

1 $(=1^2)$

1, <u>2, 3,</u> 4 $(= 2^2)$

1 and 4 are perfect square

4, <u>5, 6, 7, 8,</u> 9 $(= 3^2)$

4 and 9 are perfect square

9, <u>10, 11, 12, 13, 14, 15,</u> 16 $(= 4^2)$

9 and 16 are perfect square

16, <u>17, 18, 19, 20, 21, 22, 23, 24,</u> 25 $(= 5^2)$

16 and 25 are perfect square

Between 1^2 (=1) and 2^2 (= 4) there are two (i.e., 2×1) non square numbers 2, 3.

Between 2^2 (= 4) and 3^2 (= 9) there are four (i.e., 2×2) non square numbers 5,6,7,8.

Now, $3^2 = 9$, $4^2 = 16$

Therefore. $4^2 - 3^2 = 16 - 9 = 7$

Between 9 $(= 3^2)$ and 16 $(= 4^2)$ the numbers are 10, 11, 12, 13, 14, 15 that is, six non-square numbers which is 1 less than the difference of two squares.

We have $4^2 = 16$ and $5^2 = 25$

Therefore, $5^2 - 4^2 = 9$

Between 16 $(= 4^2)$ and 25 $(=5^2)$ the numbers are 17, 18, ….., 24 that is, eight non square numbers which is 1 less than the difference of two squares.

In general we can say that there are 2n non perfect square numbers between the squares of the numbers n and (n + 1).

3. Adding odd numbers

1 [one odd number] $= 1 = 1^2$

1 + 3 [sum of first two odd numbers] $= 4 = 2^2$

1 + 3 + 5 [sum of first three odd numbers] $= 9 = 3^2$

1 + 3 + 5 + 7 […] $= 16 = 4^2$

1 + 3 + 5 + 7 + 9 […..] $= 25 = 5^2$

1 + 3 + 5 + 7 + 9 + 11 [….] $= 36 = 6^2$

So we can say that the sum of first n odd natural numbers is n^2.

Looking at it in a different way, we can say : 'If the number is a square number, it has to be the sum of successive odd numbers starting from 1.

Consider the number 25. Successively subtract 1,3,5,7,9,….. from it

(i) 25 – 1 = 24 (ii) 24 – 3 = 21 (iii) 21 – 5 = 16

(iv) 16 – 7 = 9 (v) 9 – 9 = 0

This means, 25 = 1 + 3 + 5 + 7 + 9. Also, 25 is a perfect square.

Now consider another number 38, and again do as above.

(i) 38 – 1 = 37 (ii) 37 – 3 = 34

(iii) 34 – 5 = 29 (iv) 29 – 7 = 22

(v) 22 – 9 = 13 (vi) 13 – 11 = 2

(vii) 2 – 13 = – 11

This shows that we are not able to express 38 as the sum of consecutive odd numbers starting with 1. Also, 38 is not a perfect square.

So we can also say that if a natural number cannot be expressed as a sum of successive odd natural numbers starting with 1, then it is not a perfect square.

We can use this result to find whether a number is a perfect square or not.

4. A sum of consecutive natural numbers

Consider the following

$3^2 = 9 = 4 + 5$

Here, $4 = \text{First number} = \dfrac{3^2 - 1}{2}$;

$5 = \text{Second number} = \dfrac{3^2 + 1}{2}$

$5^2 = 25 = 12 + 13$

$7^2 = 49 = 24 + 25$

$9^2 = 81 = 40 + 41$

$11^2 = 121 = 60 + 61$

$15^2 = 225 = 112 + 113$

Note : We can express the square of any odd number as the sum of two consecutive positive integers.

5. Product of two consecutive even or odd natural numbers

$11 \times 13 = 143 = 12^2 - 1$

Also $11 \times 13 = (12 - 1) \times (12 + 1)$

Therefore, $11 \times 13 = (12 - 1) \times (12 + 1) = 12^2 - 1$

Similarly, $13 \times 15 = (14 - 1) \times (14 + 1) = 14^2 - 1$

$29 \times 31 = (30 - 1) \times (30 + 1) = 30^2 - 1$

$44 \times 46 = (45 - 1) \times (45 + 1) = 45^2 - 1$

So in general we can say that $(a + 1) \times (a - 1) = a^2 - 1$

6. Some more patterns in square numbers

Observe the squares of numbers ; 1, 11, 111 …… etc. They give a beautiful pattern :

$1^2 =$ 1

$11^2 =$ 1 2 1

$111^2 =$ 1 2 3 2 1

$1111^2 =$ 1 2 3 4 3 2 1

$11111^2 =$ 1 2 3 4 5 4 3 2 1

$11111111^2 =$ 1 2 3 4 5 6 7 8 7 6 5 4 3 2 1

Another interesting pattern.

$7^2 = 49$

$67^2 = 4489$

$667^2 = 444889$

$6667^2 = 44448889$

$66667^2 = 4444488889$

$666667^2 = 444444888889$